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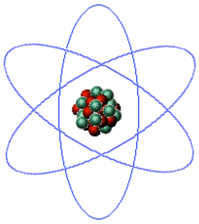
(EMAIL MAY BE BEST WAY TO CONTACT ME)

Introduction to PHY008: Atomic and Nuclear Physics



The greatest advances in technology have taken place in the last hundred years. In 1897 few would have imagined that the probing of materials at the atomic level would reveal so much. These early discoveries of atomic constituents and their structure would pave the way for semi-conductor electronics, develop key concepts in physical laws, and offer a replacement energy source for fossil fuels in the form of nuclear power. This course summarises key discoveries in early particle physics and combines historical background with the detailed physics understanding needed to fully appreciate the subject.

While lectures may impart *knowledge*, the *skills* necessary to apply this knowledge to real cases can only be obtained by practising. Examples will be given in each lecture but a huge collection of further questions, some with full solutions, are provided for problem class sessions. You are advised to try as many of these problems as you can. If you can do these questions, then the exam will be easy!!!

Recommended books

The notes are pretty complete I hope, so there are no compulsory textbooks. Any A level textbook would be helpful and the following are recommended though not essential.

Cutnell and Johnson, Physics, (Wiley, 6th edition or more recent).

Adams and Allday, Advanced Physics (Advanced Science), (Oxford), 2000.

Lectures

This topic will be covered in12 lectures in Lecture Theatre A every Thursday during term-time from 12:10am until 13:00pm.

Assessment

2 homeworks and a final examination.

Homework 1 and 2 must be submitted in E34 by 12pm on Friday 23/4/10 and 7/5/10 respectively. Late penalties will be applied.

Data Sheet

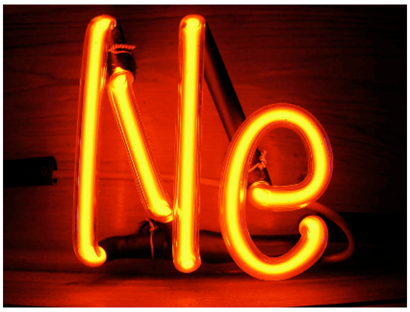
This has a number of useful formulae. It will be attached to all the physics examination papers. A copy is attached to this set of lecture notes. You should have a copy with you when doing problems or reading the notes.

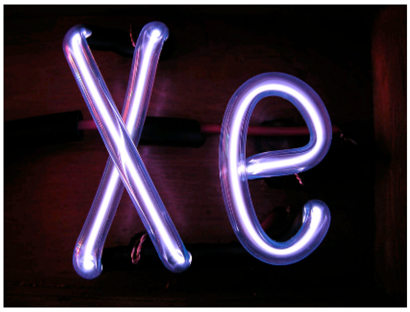
Outline Syllabus

Chapter 1: The electron and the photon

1.1 Observation of the electron

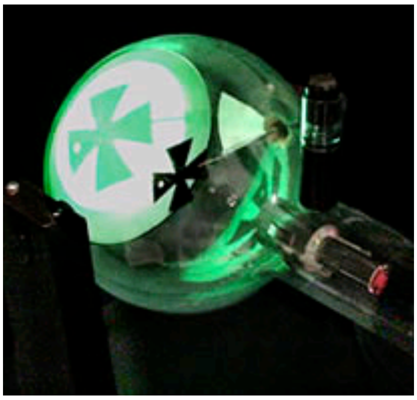
Science lecturers who travelled from town to town in the mid nineteenth century delighted audiences by showing them the ancestor of the neon sign. They took a glass tube with wires embedded in opposite ends... put a high voltage across... pumped out most of the air... and the interior of the tube would glow in lovely patterns. We now know this process as an example of the photoelectric effect (which we will meet later), but 100 years ago this was a total mystery.





Around the same time J.J. Thomson was investigating a long-standing puzzle known as "cathode rays."

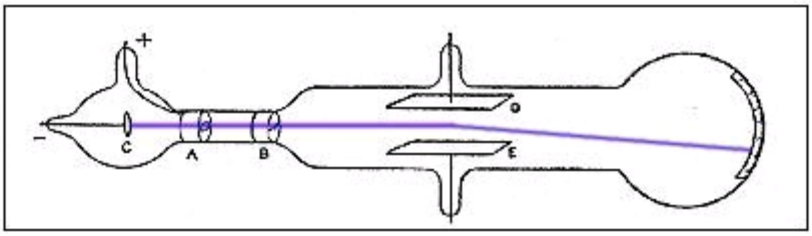
If the low pressure gas in the glass tube was replaced by a total vacuum then the glow disappeared. However, where the positive high voltage electrode passed through the glass a fluorescent glow was seen. Evidently something was being emitted by the cathode, travelling towards the anode and lighting up the fluorescent chemicals in the glass. But what could this be? Were these cathode rays similar to light waves? Or perhaps the cathode rays were some kind of particle.



In 1897 his experiments with electric fields prompted him to make a bold proposal: that these mysterious rays are streams of particles much smaller than atoms. The rays are made up of *electrons:* very small, negatively charged particles that are fundamental parts of every atom.

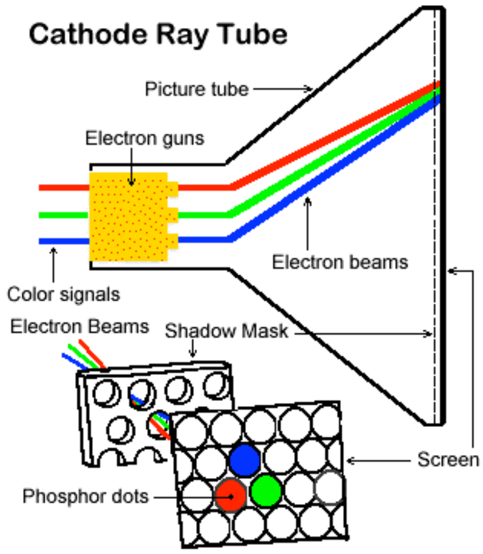
The figure shows a key experiment. ‘Cathode rays’ (later called electron beams) are observed in an evacuated glass sphere equipped with two electrodes, a cathode (negative electrode) and an anode (positive electrode). The electron gun on the right fires a beam of electrons that spread out as they travel from right to left through the vacuum towards the metallic Maltese cross which has a high positive potential. The electrons that hit the cross are stopped by the metal, but those that get past it hit a fluorescent screen at the far side of the tube which glows white when the electrons collide with it. The glow shows the shadow of the cross (anode), the brightness increasing if the potential difference between the anode and cathode is increased. The electrons had clearly originated from the cathode - hence the name.

Thomson also noted that these ‘cathode rays’ could be deflected by electric and magnetic fields. In the diagram below electrons released from the cathode plate C are accelerated along the tube to the right, towards the anode A, by a high electric potential between them. They then pass through a small hole at B in order to create a narrow beam which creates a glow on the phosphor screen on the far right hand side. If an electric field is applied between electrodes D and E, the electrons can be deflected, the amount allowing insights into their charge and mass.

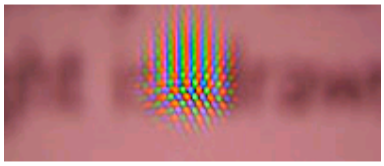


Applications: CRT Television

An old fashioned television is based around a CRT (cathode ray tube). The screen is coated on the inside surface with dots of chemicals called phosphors. When a beam of electrons hits a dot, the dot will glow. These phosphor dots are in groups of three: Red, Green, and Blue. This RGB system can then create all the other colours by combining what dots are illuminated. There are 3 signals that control the 3 electron beams in the monitor, one for each RGB colour. Each beam only touches the dots that the signal tells it to light, a shadow mask blocking the path of the other beams. All the glowing dots together make the picture that you see, the human eye blending the dots to "see" all the different colours. This is then repeated for all the other pixels on the screen by scanning across the screen, in a row 1 pixel high, from left to right, dropping down and scanning back left.



Here we see a photo of a water droplet acting as a magnifying glass on a CRT screen. See how the three colours of dots create the overall pink.



1.2 What is an electron?

An electron is an elementary subatomic particle that carries an electrical charge of -1.6×10^{-19}C. The properties of the electron have been determined by its interaction with other particles. The attractive Coulomb force between an electron and proton is what causes electrons to be bound into atoms. Current can be thought of as the flow of electrons within a conductor and is the rate of charge flow past a given point in an electric circuit, measured in Coulombs/second or Amperes. When an electron is in motion, it is deflected by external magnetic fields. When multiple electrons flow along a wire whilst in the presence of a magnetic field, the wire is deflected, giving rise to the concept of the electrical motor.

1.3 The charge/mass ratio

The charge/mass ratio is a physical quantity that is widely used to describe charged particles. But why do we single out charge and mass? We know from Newton's Laws that acceleration is proportion to the force applied to an object and inversely proportional to the mass of the object.

$$F = ma \qquad \text{and so} \qquad a = \frac{F}{m}$$

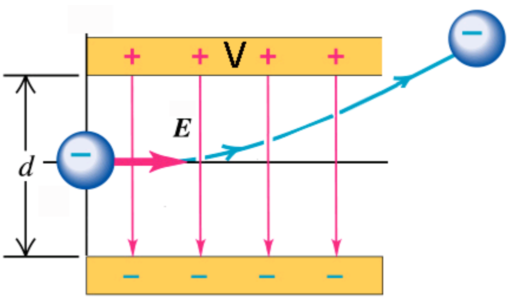
where a is the acceleration, F is the force, and m is the mass.

We know that we can put forces on charged particles such as electrons by exposing them either to an electric field or a magnetic field, or even a combination of both. The force in both cases has been shown to be proportional to the charge on the particle. The key point is therefore that two particles with the same charge/mass ratio will move in the same path in a vacuum when subjected to the same electric and magnetic fields (see later for applications: mass spectrometer).

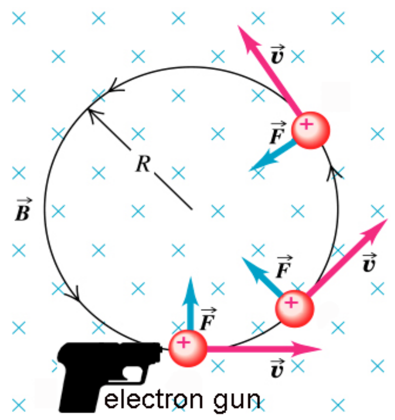
Let's remind ourselves of the forces on a particle of mass m and charge q:

i) in an electric field E, the force acts towards the opposite polarity as shown below for a stream of electrons passing from left to right. The path is therefore parabolic like a projectile falling under gravity for the period it feels the force.

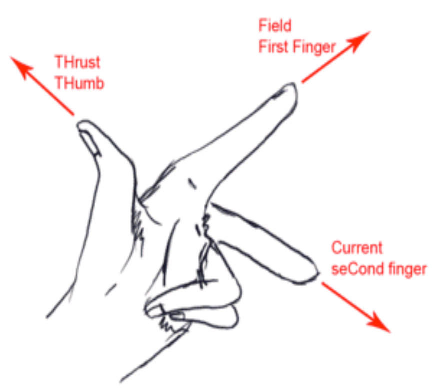
$$F_{electric} = qE \qquad \text{where} \qquad E = \frac{V}{d}$$



ii) in a magnetic field *B,* where *B* and the particles velocity *v* are at right angles, the force is also at right angles to both. The particle therefore travels in a circle since the force is always at right angles to the motion.



$$F_{magnetic} = Bqv \quad \text{when } B, v, \text{ and } F \text{ are orthogonal.}$$



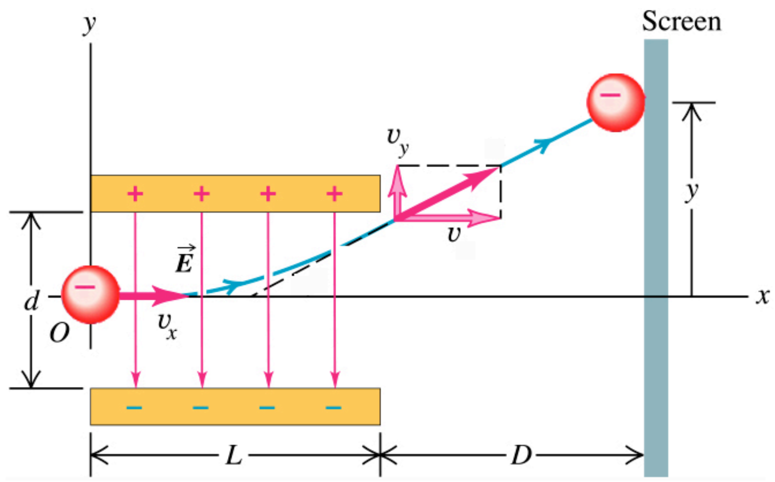
The direction of the field is represented by either an × to indicate it is passing into the paper or a • to show it is coming out of the paper.

The direction of the force on the particle is determined by Fleming's Left Hand Rule. Here the thu**M**b represents the direction of **M**otion due to the force, the **F**irst finger represents the direction of the magnetic **F**ield and the se**C**ond finger represents the direction of **C**urrent flow (defined as the flow of positive charge).

1.4 Thomson's experiment to find the charge/mass ratio

In 1896 Thomson used a combination of electric and magnetic fields to measure the charge/mass ratio of the electron. Let's look at how he did this by building up his experiment in stages. (This will serve as an excellent demonstration of the effects of electric and magnetic fields on charged particles, and as such is a critical section of the course).

First let's look at the effect of just the electric field. The arrow shows a beam of electrons passing between two parallel plates, all in vacuum.



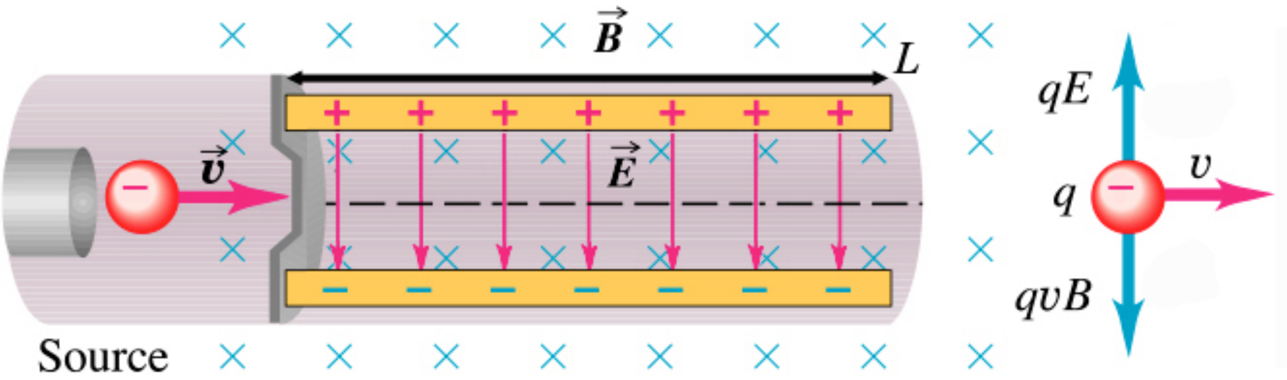
The force due to the *E* field is $F_{electric} = qE$, so the vertical acceleration of the electron is $a = \frac{qE}{m}$.

This force acts only while the electron is between the plates, a time $t = \frac{L}{v}$ where *v* is the velocity in the beam direction. So the vertical velocity is $v_{vertical} = at = \frac{qE}{m}\frac{L}{v}$ when the electron exits at $x = L$.

After that this vertical velocity does not change, and the electron carries on towards a screen with constant horizontal velocity. If the distance to the screen is D, (and D>>L), then the time T taken for the electron to travel to the screen is given by $T = \frac{D}{v}$. In this time the electron will have also had a vertical velocity $v_{vertical}$ and so we can say that when the electron hits the screen it has a vertical displacement *y* given by :-

$$y = Tv_{vertical} = \frac{D}{v}\frac{qEL}{mv} = \frac{qDEL}{mv^2}$$

This expression has *q/m* (the thing we want) in it, but although we can measure *y*, *D*, *E*, and *L*, we don't know *v*. So we then apply a magnetic field at right angles (into paper) so that the electron experiences a force in the opposite direction to that from the electric field, increasing B until $y = 0$ and the forces therefore balance i.e. $Bqv = qE$ as shown in the figure below.

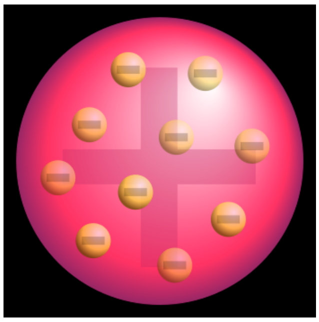


So now $v = \frac{E}{B}$ and after substituting back into the expression for y we find:-

$$y = \frac{qDEL}{mv^2} = \frac{qDELB^2}{mE^2} = \frac{qDLB^2}{mE} \quad \text{and therefore that} \quad \boxed{\frac{q}{m} = \frac{yE}{DLB^2}}$$

Thomson showed that $\frac{q}{m}$ = -1.76×10^{11} C Kg^{-1} where coulombs (C) are the unit of charge.

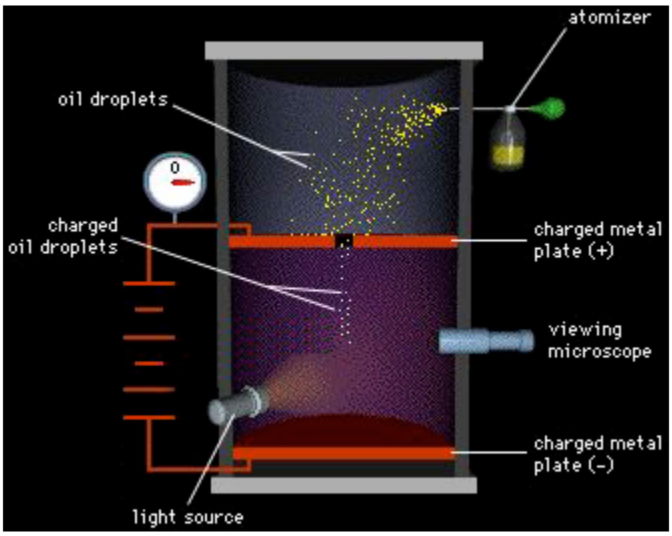
Thomson didn't know the absolute quantities of *q* and *m*, just their ratio. He then repeated the experiment for hydrogen ions, finding the value of hydrogen's *q/m* ratio, which was much smaller than that for the electron. So he reasoned that either the *q* of the electron was bigger than the *q* of hydrogen or the *m* of electron was smaller then the *m* of hydrogen. From 1865 scientists had been able to calculate the mass of a hydrogen atom. The electron on the other hand seemed to pass from a cathode to an anode through a vacuum with apparently no transfer of mass from one electrode to the other. He therefore came to the conclusion that the mass of the electron was the one which was very small.



This was a bold statement as it had been believed for millennia that the atom (e.g. hydrogen) was the smallest building block in the universe. Based on this result Thomson formed the hypothesis that the atom was in fact divisible, and consisted of some smaller particles. In 1904 he created the idea of the atom as a ‘plum pudding’.

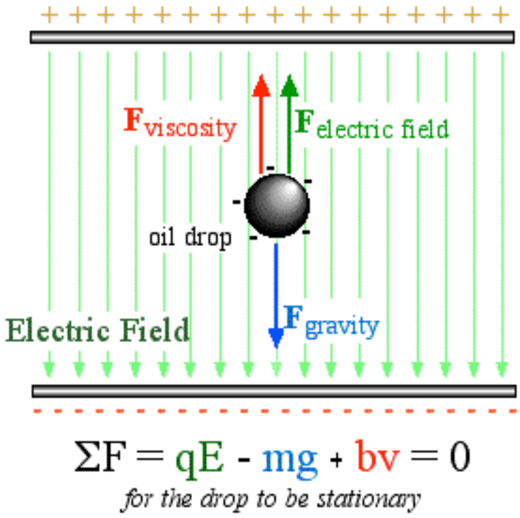
In this model, (before the discovery of the atomic nucleus), the atom was composed of electrons, surrounded by a soup of positive charge to balance the electron's negative charge, like negatively-charged "plums" surrounded by positively-charged "pudding". This concept existed until 1911, when Ernest Rutherford built his own model of the atom - the "planetary" one (see later in Chapter 2).

1.5 Millikan’s experiment to find the charge of the electron



In 1909 Millikan measured the charge of the electron by carefully balancing the gravitational and electric forces on tiny charged droplets of oil suspended between two metal electrodes, noting that very fine sprays of oil produced charged droplets which he suggested were carrying the charge *q* of a few electrons. A light source made it possible to look at the motion of the droplets through a microscope which was placed in the chamber. In the lens of the microscope there were two lines, whose separation was measured beforehand. The velocity of the droplets could then be found by timing the droplets as they travelled the predetermined distance between the lines. Knowing the electric field, the charge on the oil droplet could be determined. Repeating the experiment for many droplets, he found that the values measured were always multiples of the same number. He interpreted this as the charge on a single electron: 1.6×10^{-19} coulombs. Let's look at the experiment in detail.

Initially the oil drops are allowed to fall between the plates with the electric field turned off. They very quickly reach a terminal velocity at which point the gravitational force is balanced by the frictional force due to air resistance (blue arrow balances red arrow in figure). A single oil droplet is selected and its terminal velocity v_1 is measured. The drag force acting on the drop can be worked out using Stokes' law: $F_{drag} = 6\pi r \eta v_1$ where v_1 is the terminal velocity of the falling drop, η is the viscosity of the air, and r is the radius of the drop.



The volume of the droplet is simply $V = \frac{4}{3}\pi r^3$ and the density is $\rho = \frac{M}{V}$ and so the mass of the droplet is $M = \rho V = \frac{4}{3}\pi r^3 \rho$ and the weight is $W = Mg = \frac{4}{3}\pi r^3 \rho g$.

Since $F_{drag} = Mg$ when the droplet is at terminal velocity we can write: $F_{drag} = 6\pi r \eta v_1 = \frac{4}{3}\pi r^3 \rho g$

and by rearranging we can say: $\boxed{r^2 = \frac{9\eta v_1}{2\rho g}}$. Once r is calculated, W can easily be determined.

Now the electric field is turned on and the droplet experiences a force due to the electric field of $F_{electric} = qE$ where q is the charge on the oil droplet and E is the electric field between the plates. The electric field is then adjusted until the oil droplet remains steady, meaning that the force due to the electric field is equal to the weight of the droplet. So $F_{electric} = qE = W$ and therefore $\boxed{q = \frac{W}{E}}$.

Since the weight of the droplet has already been found, the charge can be calculated.

Doing this for a huge number of droplets he discovered that the values calculated for the charge q on each droplet were all integer multiples of -1.60×10^{-19} C. He therefore stated that this was the charge of the electron and that the droplets contained multiple numbers of electrons, i.e. $q = n\times$(electron charge) with n an integer number of electrons.

Using Thomson's result for the charge/mass ratio, the electron mass was then $m_e = 9.1\times10^{-31}$ Kg.

1.6 Electron-volts

This is such a tiny topic but it is also one of the most important ones. You will use electron-volts in two main ways; (i) as a convenient unit of energy, and (ii) to calculate the velocity of a charged particle passing through an electric field.

Definition: The change in energy E of a charged particle q as it moves through a potential difference V is given by $E = qV$. 1 electron-volt $=1eV=1.6\times10^{-19}\times1 = 1.6\times10^{-19}$ J.

(i) In particle physics we often deal with very small energies and it is often therefore more convenient to refer to 3.2×10^{-19} J as 2 eV for example.

(ii) Imagine a particle of mass m and charge q is accelerated from rest through a voltage V. The energy given to the electron is therefore qV . If all this energy is converted into motion then this will be equal to the final kinetic energy of the particle. So $\frac{1}{2}mv^2 = qV$ and the velocity of the electron is $v = \sqrt{\frac{2qV}{m}}$ where v is in metres per second, V is in volts, q is in coulombs, and m is in kilograms.

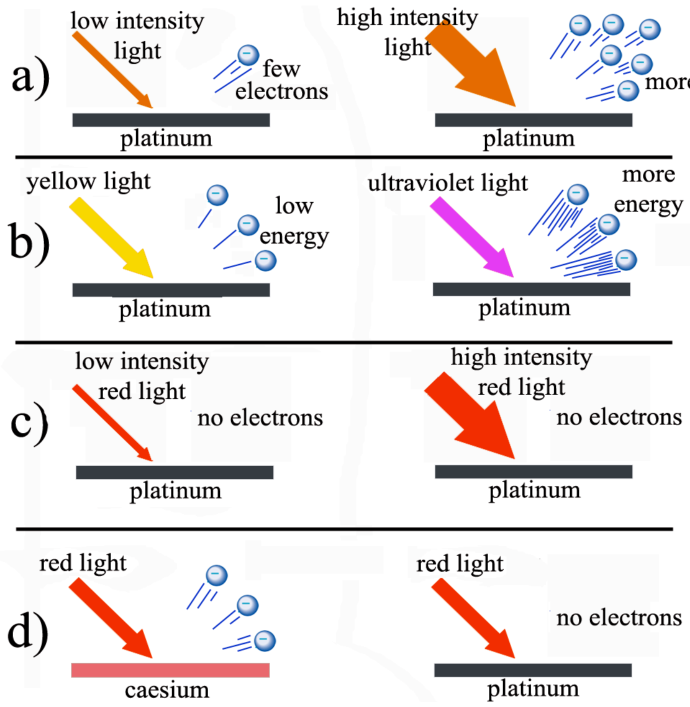
1.7 Photoelectric effect - electrons and light.

In order to continue our journey charting the understanding of atomic structure we now need to look at the evidence which prompted scientists to improve on the plum pudding concept. At the end of the 19th century scientist noted that when some metals are illuminated by a strong source of light, electrons are emitted from the surface of the metal. This is called the photoelectric effect and the electrons are called photoelectrons to show how were created (although they are no different from normal electrons). But the scientists were confused. The kinetic energy of the ejected photoelectrons was found to depend on the frequency of light and the type of metal used and not, as expected, on the brightness of the light source. Intuitively we imagine that increasing the amplitude of the incident waves would cause the kinetic energy of the ejected photoelectrons to increase, in the same way that a large amplitude water wave would impart more kinetic energy to pebbles on a beach. However although the amplitude of the incident light had no effect on the kinetic energy of the ejected photoelectrons, greater kinetic energy could be imparted to the ejected photoelectrons by increasing the frequency of the incident light!!! This initially made no sense. The figure below shows the results observed.

Einstein offered an explanation in 1905 by proposing that light was not simply a wave but rather made up of tiny quanta or packets of energy, the energy of a single photon proportional to the frequency of light.

$$E_{photon} = hf$$

where h is known as Plank's constant (6.626×10^{-34}Js) and f is the frequency of the light in Hz.



Einstein proposed that there was a minimum energy E_0 required to release a photoelectron from a metal. He called E_0 the work function and suggested that this value was a constant for a particular metal, but was different for different metals. When a photon is absorbed within a metal, some of the photon's energy will be used up in freeing the photoelectron from the metal, and if there is any energy remaining, then this will appear as kinetic energy of the ejected photoelectron.

$$E_{kinetic\,energy} = hf - E_0$$

Key points of the photoelectric effect:-

a) The fact that the number of photoelectrons increases with the intensity of the light is explained by each photon liberating exactly one photoelectron. A higher intensity of light implies that more photons are present and so more photoelectrons are ejected.

b) The fact that the maximum kinetic energy of the photoelectrons depends on the frequency of light is explained because photons corresponding to light of a higher frequency carry more energy. So after E_0 has been used up, there is more energy left over to appear as kinetic energy of the photoelectron.

c) The fact that there is a lower limit for the frequency of light, below which no photoelectrons are emitted is due to the fact that since the minimum energy required to eject an electron is E_0, then the minimum frequency of light needed to do this is $E_0 = hf$.

d) The fact that for the same light source some metals eject photoelectrons and other do not is due to the different work function E_0 of different metals. Metals, such as caesium, with a lower work function need less of the photon's energy to release a photoelectron and so are more likely to exhibit photoelectron emission.

1.7.1 Let's look at an example.

Imagine a ray of green light of wavelength λ = 530 nm incident on a metal with a work function of 1.1eV. What is the kinetic energy given to a photoelectron ejected from this metal? The frequency of green light is found using $c = f.\lambda$ where c is the velocity of light, so $f = \frac{3\times10^8}{530\times10^{-9}} = 5.66\times10^{14}\,\text{Hz}$. Einstein said this ray was made up of photons each of energy $E_{photon} = hf = 6.626\times10^{-34}\times5.66\times10^{14} = 3.54\times10^{-19}\,\text{J}$.

So our green photons each have energy $\frac{3.54\times10^{-19}}{1.6\times10^{-19}} = 2.3\,eV$ (electron volts).

The energy required to release an electron from the metal is 1.1 eV. The remaining energy of 2.3 - 1.1 = 1.2 eV is given to the ejected photoelectron in the form of kinetic energy.

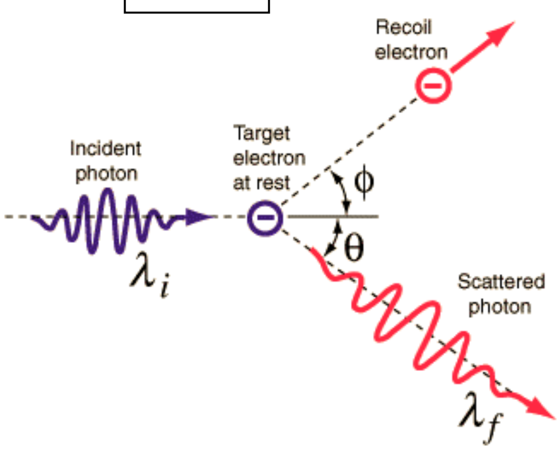
What is the lowest wavelength of light that can release an electron from this metal? The energy required to release an electron is 1.1 eV. The photon energy is $E_{photon} = hf$ and so the minimum energy is

$E_{photon} = 1.1\,\text{eV}$. Therefore the lowest frequency is $f = \frac{E_{photon}}{h} = \frac{1.1\times1.6\times10^{-19}}{6.626\times10^{-34}} = 2.81\times10^{14}\,Hz$. This corresponds to a wavelength of $\lambda = \frac{3\times10^8}{2.81\times10^{14}} = 1.067\,\mu\text{m}$, in the near infra red.

Einstein had proposed that despite all the evidence that light is a wave, it also has particle-like properties. There was previously no connection between the energy of a light wave and its frequency. Later in 1923 de Broglie, influenced by Einstein's work, suggested that particles such as electrons could exhibit wave-like qualities, and waves such as light could exhibit particle-like behaviour. He called this wave-particle duality.

1.8 The Compton effect – particle nature of light

In the last section we saw how Einstein imagined waves of light to be in fact tiny particles called photons in order to explain the photoelectric effect, and how he linked the energy of the photon to the frequency of the wave with the equation $E_{photon} = hf$. Einstein's most famous equation relates energy to mass and is written $E_{photon} = mc^2$ where m is the mass of the photon. The momentum p of any particle is equal to its mass multiplied by its velocity i.e. $p = mc$ and since $E_{photon} = hf = mc^2$, we can rewrite this as $E_{photon} = hf = pc$ and so momentum of a wave can be written as $\boxed{p = \frac{h}{\lambda}}$. This is called the de Broglie equation and we will look at it in more detail in the next section.



Many scientists questioned this interpretation. What was needed was an experiment to demonstrate the particle nature of photons of light. In 1923 Arthur Compton did this by setting up a collision between X-ray photons and electrons. He chose these carefully as we know from traditional classical physics that in order to observe a momentum change between two particles, it is best for their masses to be similar (otherwise the particle with the far higher momentum will hardly be deflected).

The experiment showed that the X-ray photons and electrons behaved exactly like ball bearings colliding on a table top. Because the electron was scattered, the photon must have transferred both momentum and kinetic energy to it. This can only be explained by assuming that photons have momentum. But he observed something else. Before the collision the photon had one wavelength and after the collision its wavelength had increased. Clearly the electron had been given energy, conservation of energy indicating that the scattered photon must therefore have lower energy than prior to the collision. The increase in wavelength, corresponding to a drop in frequency, could then only be explained by assuming the $E_{photon} = hf$ relationship.

The effect is important because it demonstrates that light cannot be explained purely as a wave phenomenon, the classical theory of an electromagnetic wave scattered by charged particles unable to explain any shift in wavelength. Light must behave as if it consists of particles in order to explain Compton scattering.

1.8.1 Let's look at an example.

Let's imagine that we collide a photon of green light (λ = 530 nm) with an electron. What is the momentum of a photon of green light?

First let's calculate the momentum of the photon using $p = \frac{h}{\lambda}$. We always use SI units so mass is in kg, distance is in m, time is in s, energy is in joules. So $p = \frac{h}{\lambda} = \frac{6.626\times10^{-34}\,Js}{530\times10^{-9}\,m} = 1.25\times10^{-27}\,\text{kgms}^{-1}$.

If we would like the electron in this case to have an equal momentum, what voltage must we accelerate the electron through in order to achieve this?

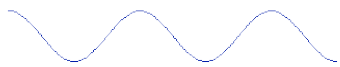
We know that an electron has a mass of 9.1×10^{-31} kg so for it to have equal momentum to the photon, it needs to travel at a velocity of $v = \frac{p}{m_e} = \frac{1.25\times10^{-27}}{9.1\times10^{-31}} = 1373\,\text{ms}^{-1}$. We know from section 1.6 that we can produce such electrons by accelerating them through a voltage V given by $V = \frac{m_e v^2}{2e}$ so voltage required is $V = \frac{m_e v^2}{2e} = \frac{9.1\times10^{-31}\times(1373)^2}{2\times1.6\times10^{-19}} = 5.37\,\mu V$.

This experiment finally convinced scientists that light had both wave-like (diffraction, interference) and particle-like collisions which conserved both energy and momentum.

1.9 de Broglie and matter waves

In 1923 de Broglie argued that since light can display both wave and particle-like behaviour, then perhaps matter (protons, neutrons, and electrons) could also be thought of as both particles and waves too. He called this wave-particle duality.

Normal waves (blue) look like this with no beginning and no end.



Particles on the other hand look like this.

A photon can be thought of as a wave packet (red), having both wave-like properties and also the single position and size we associate with a particle.

To test this concept he needed to try to get matter to demonstrate wave-like properties such as diffraction or interference for example using a double slit apparatus. de Broglie said that since $p = \frac{h}{\lambda}$ works for light, why should it not work 'in reverse' for matter? Now previously for light we defined the energy of a photon using Einstein's $E_{photon} = mc^2$ which applies to photons moving at the speed of light where special relativity needs to be taken into account.

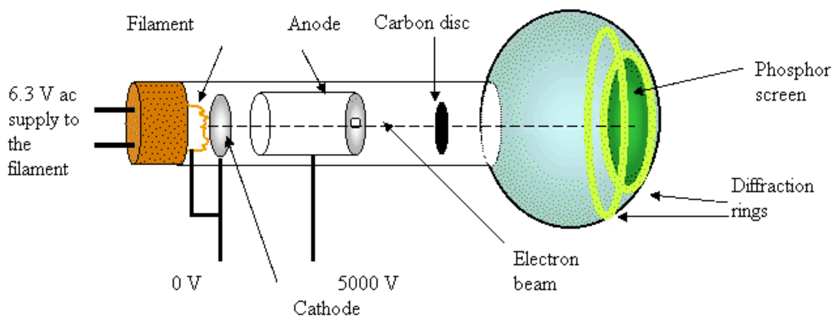
Matter does not generally move at the speed of light and so its energy can simply be given by the standard form of kinetic energy $E = \frac{mu^2}{2}$ and momentum $p = mu$ where m is the mass and u is the velocity.

So $u^2 = \frac{2E}{m}$ and therefore $u = \sqrt{\frac{2E}{m}}$ so $p = mu = m\left(\sqrt{\frac{2E}{m}}\right) = \sqrt{2mE}$.

From standard double slit experiments using light we know that we only see clear evidence of interference if the spacing of the slits or grating is about the same as the wavelength λ of light. It is therefore sensible to select matter waves of similar wavelength to the slit spacing. This caused him a great deal of difficultly. Let's see why using an example.

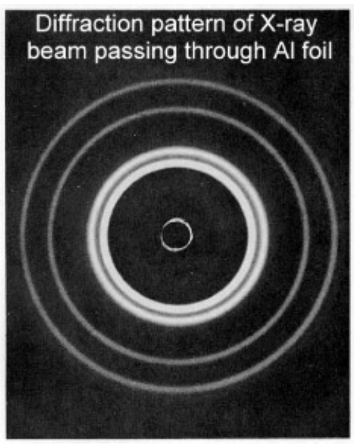
Consider an electron moving with velocity $u = 4.68\times10^{7}$ m s^{-1}. Since we know $m_e = 9.1\times10^{-31}$ kg then we can say that it has momentum $p = mu = 9.1\times10^{-31}\times4.68\times10^{7} = 4.27\times10^{-23}\,kgms^{-1}$. Its wavelength according to the de Broglie equation $\lambda = \frac{h}{p} = \frac{6.626\times10^{-34}}{4.27\times10^{-23}} = 1.55\times10^{-11}\,\text{m}$. This is a small length in terms of the world around us. Even atoms in a crystal are typically no less than 2×10^{-10} m apart, over 10 times more than this wavelength. If we select a very thin crystal of atoms at this spacing (such as a thin disc of carbon) and shine a light wave at it, the regular spacing of atoms forms a kind of diffraction grating producing lines or spots on a distant screen. Suppose instead we fired a beam of electrons at this crystal what would we see?

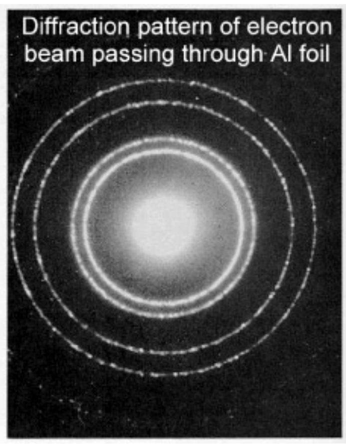
Since electrons are thought of as being small particles intuitively we would expect them to bounce off the crystal. However the result is a diffraction pattern of very similar appearance to that expected from a light source of similar wavelength, as shown in the figure below.



Electrons therefore have wave-like properties, as well as particle-like ones. So why do we not see diffraction effects when a person collides with a metal grating?! Well a person is made up of a huge number of particles. Even a single hydrogen atom is 1837 × m_e. Remember that in order to see diffraction effects the wavelength should be approximately the same as the slit separation. But $\lambda = \frac{h}{p}$ and since the momentum of a person is far far greater than the momentum of an electron, the wavelength λ of the matter wave is incredibly small and certainly far smaller than the smallest slit separations. The only way to detect diffraction from matter waves is to slow the particles down and reduce their mass. Even so, a bowling ball for example would require sizes of slits of the order of 10^{-34} m or so, which is far beyond present-day technology. This is why we are not aware of matter waves in the everyday world and why people don't diffract through chairs when they sit down.

While de Broglie matter waves were difficult to accept after centuries of thinking of particles as solid things with definite size and position, electron waves had been confirmed in the laboratory by running electron beams through slits and demonstrating that interference patterns are formed in a similar way to those produced using X-ray photons.





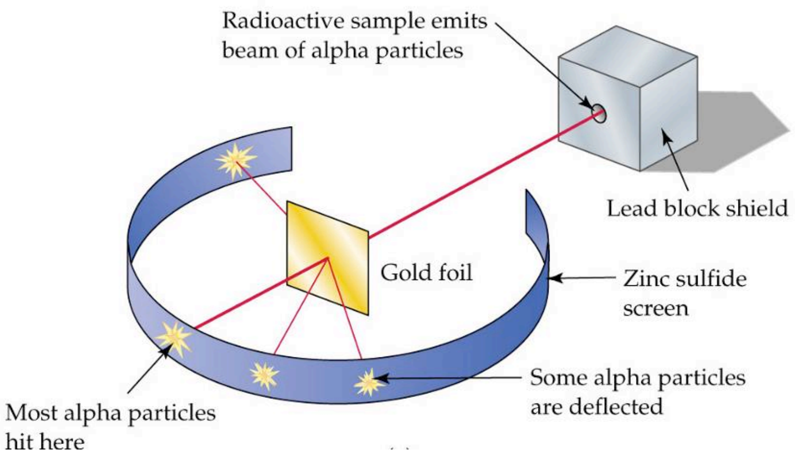
Chapter 2: The atom

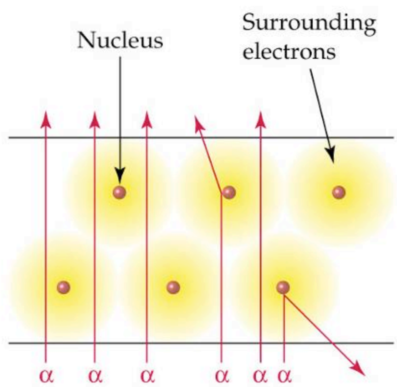
Now that we have discussed the photoelectric effect and wave particle duality, let's get back to the way our concept of the atom has evolved. In section 1.4 we learned that in 1904, Thomson considered atomic structure to be represented by a 'plum pudding' in so far as the atom was made up of negatively charged electrons surrounded by a soup of positive charge to keep the overall charge neutral.

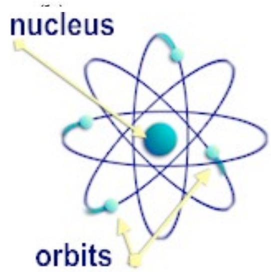
2.1 Rutherford and the planetary system of the atom

Atoms are electrically neutral, so if they contain electrons they must also contain some positive charge, but where? Thomson assumed that this was uniformly distributed like a soup throughout the atom but this was only conjecture. To find out, in 1909 Rutherford fired positively charged helium (He^{2+}) ions with high energy at gold atoms in a thin gold film. Based on the theory that positive and negative charges were spread evenly within the atom and that therefore only weak electric forces would be exerted on the high energy ions passing through the thin foil, he expected to find that most of the ions travelled straight through the foil with little deviation.

What he found, to great surprise, was that whilst most passed straight through the foil, a small percentage (about 1 in 10000) were deflected at very large angles and some even bounced back toward the ion source. Because helium ions are about 8000 times the mass of an electron and impacted the foil at very high velocities, it was clear that very strong forces were necessary to deflect these particles. (Imagine firing bullets at soup. Even if just one ricocheted back it would be surprising!!)







The He^{2+} positive ions had clearly been repelled by an incredibly large positive charge within the atom, this charge concentrated in a dense region also containing most of the mass. This work led in 1913 to Rutherford declaring the atom to contain a very small nucleus of high positive charge (equal to the number of electrons in order to maintain neutrality) and to be similar to the 'solar-system-like' model, in which a positively charged nucleus is surrounded by an equal number of electrons in orbital shells.

From purely energetic considerations of how far helium ions of positive charge and known velocity would be able to penetrate toward the central positive charge of the gold nucleus, Rutherford was able to calculate that the radius of the gold nucleus would need to be less than 3.4×10^{-14} metres (the modern value is only about a fifth of this). The radius of the entire gold atom was known at the time from diffraction measurements to be 10^{-10} metres or so. This then implied that the diameter of the nucleus, containing all the positive charge and almost all of the mass, was less than 1/3000th the diameter of the atom.

Detailed analysis of scattering data for various elements has since found the diameter of the nucleus to range from 2 to 9×10^{-15} m from hydrogen to uranium, and the diameter of the nucleus to be almost 10^{-5} the diameter of the atom. We now also know the nucleus contains both protons (positively charged) and neutrons (electrically neutral) particles of almost the same mass. Scientists at this time had no idea about the existence of neutrons.

2.2 The hydrogen atom and the Bohr model

It was soon realized that the Rutherford planetary model, although very attractive, posed a major problem. It was well known that electrons travelling in circular orbits emitted radiation, thus losing energy causing their trajectories to collapse towards the centre of the orbit. It was therefore clear in the Rutherford model that these electrons described by the planetary model would quickly collapse rendering the atom unstable.

In order to save the main characteristics of the model Bohr assumed that it would be necessary to modify the laws governing the motion of the electrons within the atom. As a consequence, the Rutherford model was soon superseded by the Bohr model, which used some of the early quantum mechanical results to give positional structure to the behaviour of the orbiting electrons, restricting them to certain specific circular orbits.

Hydrogen is the simplest atom: one electron orbits around one proton, their electrostatic attraction to one another balanced by the centrifugal force due to the orbit.

We know that centripetal acceleration of a rotating body is $a = \frac{v^2}{r}$ where v is velocity and r the radius.

Since the centrifugal force $F_{centrifugal} = ma$ then $F_{centrifugal} = \frac{mv^2}{r}$.

The electrostatic attraction between two charges q and q' is given by $\boxed{F_{electrostatic} = \frac{q\,q'}{4\pi\,\varepsilon_0\,r^2}}$

In our case the charges on the proton and electron are equal and opposite so $F_{electrostatic} = \frac{-e^2}{4\pi\,\varepsilon_0\,r^2}$

Since we know that the radius of the electron orbit is constant, these forces must balance and so we can say that: $F_{electrostatic} = F_{centrifugal}$ so $\frac{mv^2}{r} = \frac{e^2}{4\pi\,\varepsilon_0\,r^2}$ and therefore $\boxed{mv^2 = \frac{e^2}{4\pi\,\varepsilon_0\,r} \quad \text{equation 2.1.}}$

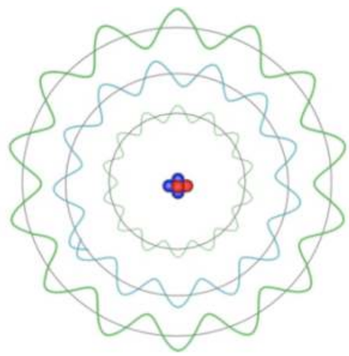
The total energy of the electron is the sum of its kinetic energy as it circles the nucleus and its electrical potential energy due to its height above the nucleus. (This is very similar to the total energy of a satellite orbiting above the earth being the sum of its kinetic energy and gravitational potential energy).

So $E_{total} = K.E. + P.E.$ We know that $K.E. = \frac{mv^2}{2}$ and so from the above $KE = \frac{mv^2}{2} = \frac{e^2}{8\pi\,\varepsilon_0\,r}$.

The electric potential energy is defined as $E_{potential} = \frac{q\,q'}{4\pi\,\varepsilon_0\,r}$ so in this case $\boxed{E_{potential} = \frac{-e^2}{4\pi\,\varepsilon_0\,r}}$

Total energy is $E_{total} = K.E. + P.E.$ so $E_{total} = \frac{e^2}{8\pi\,\varepsilon_0\,r} + \frac{-e^2}{4\pi\,\varepsilon_0\,r}$ and so $\boxed{E_{total} = \frac{-e^2}{8\pi\,\varepsilon_0\,r} \quad \text{equation 2.2.}}$

So now we know the total energy of the electron orbiting around a hydrogen nucleus. But what determines the radius of the electron orbit around the atom? Remember the de Broglie matter waves mentioned earlier and how a particle such as an electron can be thought of as a wave. Basic maths tells us that the circumference of a circular orbit is $2\pi r$. To get a matter wave into this length requires an integer number of wavelengths as shown in the diagram. (Think of a wavy piece of string: this is the only way that we can join up the ends to form a smooth shape).



Stating this mathematically we say that: $2\pi\,r = n\lambda$

The de Broglie wavelength of the electron was shown earlier to be $\lambda = \frac{h}{mv}$ and so $2\pi\,r = \frac{n\,h}{mv}$

Therefore $mv = \dfrac{nh}{2\pi r}$ and so $m^2v^2 = \dfrac{n^2h^2}{4\pi^2 r^2}$ and thus $\boxed{mv^2 = \dfrac{n^2h^2}{4\pi^2 r^2 m} \quad \text{equation 2.3}}$

Equating equations 2.1 and 2.3 we have: $\dfrac{n^2h^2}{4\pi^2 r^2 m} = \dfrac{e^2}{4\pi\varepsilon_0 r}$ and so $\boxed{r = \dfrac{n^2h^2\varepsilon_0}{\pi m e^2} = 5.26\times10^{-11}\,n^2 \text{ metres.}}$

***n* has to be an integer** as it refers to the integer number of de Broglie wavelengths. '*r'* is known as the Bohr radius and is the closest the electron can get to the nucleus when $n = 1$.

Finally inserting the expression for r into equation 2.2 we can write an expression for the total energy of the electron: $E_{total} = \dfrac{-e^2}{8\pi\varepsilon_0 r} = \dfrac{-e^4 m}{8\varepsilon_0^2 h^2 n^2} = \dfrac{-2.18\times10^{-18}}{n^2}$ joules. Or in different units $\boxed{E_{total} = \dfrac{-13.6}{n^2} \text{ eV.}}$

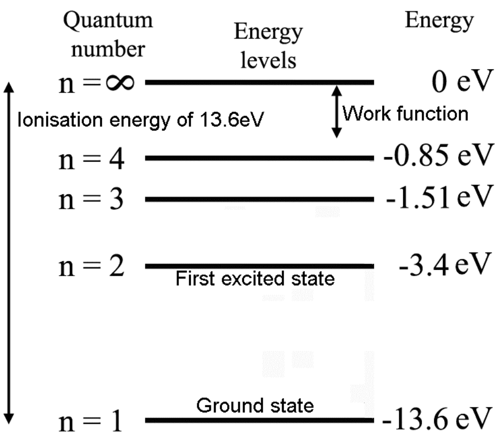
We can see that for hydrogen the electron radius and also the electron energy levels can only take specific values determined by the integer *n* which is called the quantum number. This explanation is consistent with the observations of line spectra emitted from excited gases mentioned in section 1.1.

<u>2.3 Testing the model - spectral lines</u>

The Bohr model states that electrons orbiting a proton in a hydrogen atom can only have specific energies determined by the value of the integer *n*.

Convention says that these energies are negative because the electron is bound to the atom whereas an electron that is just free has an energy = 0 (*n* = infinity).

An energy level diagram represents this pictorially. The lowest level (*n* = 1) is known as the ground state, whilst others are known as excited states. The electron must occupy one of these energy levels. By supplying energy to the electron we can excite them into higher energy levels.



n	E_{total} (eV)
1	$E_1 = \dfrac{-13.6}{1^2} = -13.6$
2	$E_2 = \dfrac{-13.6}{2^2} = -3.4$
3	$E_3 = \dfrac{-13.6}{3^2} = -1.51$
4	$E_4 = \dfrac{-13.6}{4^2} = -0.85$
∞	$E_\infty = \dfrac{-13.6}{\infty^2} = 0$

The work function is the minimum energy needed to remove an electron from a solid to a point immediately outside the solid surface. The value of the work function for metals is lower than the ionisation energy because when metal atoms are grouped together, a 'sea' of free electrons (located in the conduction band) is created which allows conduction of electricity. The work function is the energy needed to liberate one of these electrons, whereas the ionisation energy is defined as the energy required to remove an electron from one of the energy levels n = 1,2,3, etc (in the valence band) in a single atom.

<u>2.3.1 Absorption of photons</u>

The electron, orbiting in the hydrogen atom will generally be in its ground state. If we shine white light (i.e. photons of all frequencies) through the gas, the photons can only be absorbed if the photon energy is just that required to lift an electron from the ground state to an excited one.

$E_{photon} = hf$ where *h* is Plank's constant (6.626×10^{-34}Js) and *f* is the frequency of the light.

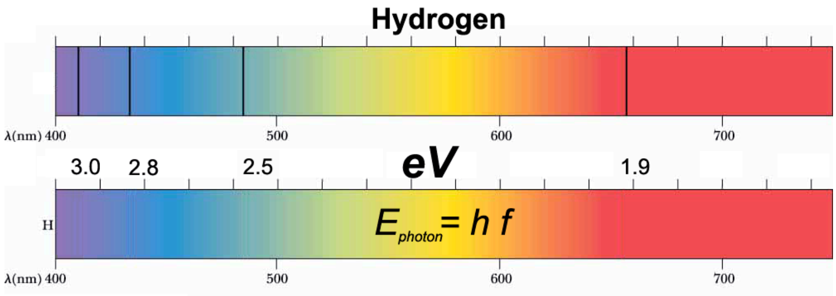
For hydrogen, the energy of the electron in its ground state *n = 1* is : $E_1 = -13.6$ eV. The energy then required to transfer it to a higher energy level n is given by: $\Delta E = E_n - E_1 = \left(\dfrac{-13.6}{n^2}\right) - (-13.6)$

So $\Delta E = 13.6 - \left(\frac{13.6}{n^2}\right) = 13.6\left(1 - \frac{1}{n^2}\right)$ in eV.

Remembering that this must equal the energy of the photon we can say: $E_{photon} = 13.6\left(1 - \frac{1}{n^2}\right)$ in eV

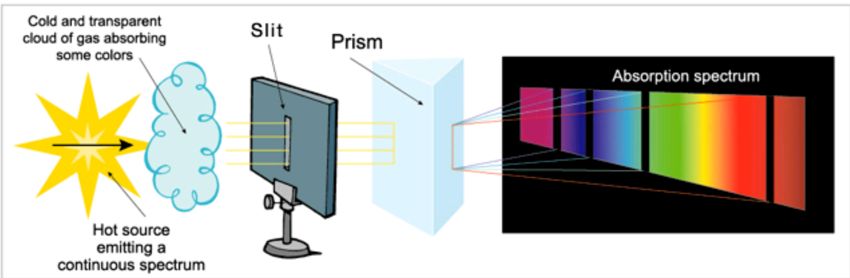
or frequency $f_{photon} = 3.3 \times 10^{15}\left(\frac{n^2 - 1}{n^2}\right)$ in Hz.

At these energies or frequencies we see dark lines (missing photons) across the otherwise white light output spectrum as shown in the diagram below. The missing photons have been absorbed by the hydrogen gas, their energy used to promote electrons in the gas to higher energy levels.



Looking at the continuous spectrum and comparing it with the hydrogen absorption spectrum we see that the four absorption lines are at 1.9, 2.5, 2.8, 3.0eV. These energies correspond to the energy gaps between excited states. For example, you should be able to see that the 1.9 eV absorption line is related to the transition between the n = 2 and n = 3 excited states, and that the 2.5 eV line is related to the transition between the n = 2 and n = 4 states. (Although there are only four lines in the visible region of the electromagnetic spectrum, there are many more elsewhere).

This provides a characteristic spectrum which can be used for example to identify gases around planets or the composition of stars. The central region of a star tends to radiate most of the light, whilst the upper layers act like the low density gas through which this light passes. Stars contain all of nature's chemical elements and as a result, the spectrum of a star displays an extraordinary mixture of absorption lines. Over 100,000 absorption lines are visible in the Sun's spectrum.



2.3.2. Emission of photons

In order for a gas such as neon or hydrogen to emit light, we must provide the orbiting electrons within the atoms with energy. This can be achieved via electrical discharge, the electrons within the gas being excited from the ground state into excited states. Eventually the atom radiates a photon to lower its energy, the energy loss taking the electron to one of the lower allowed energy states. The frequency of emitted light is then determined by the difference in energy between the states.

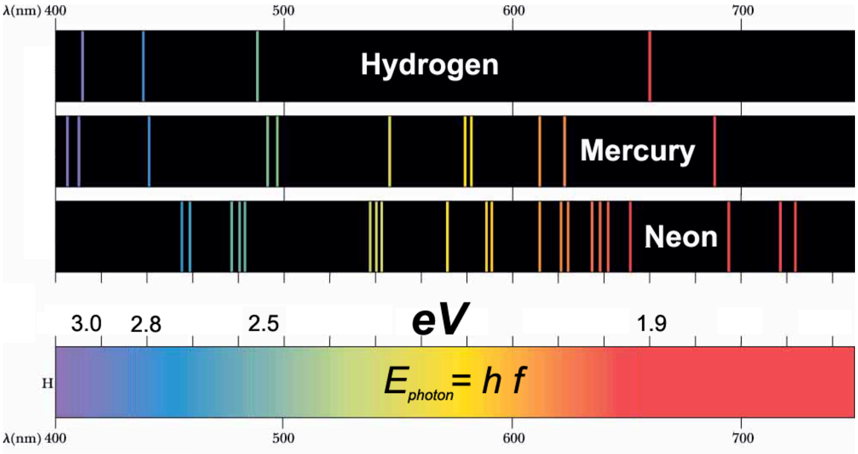
If the excited state has a quantum number of p and the electron subsequently drops to quantum number q then the change in energy is given by: $\Delta E = E_p - E_q = \left(\frac{-13.6}{p^2}\right) - \left(\frac{-13.6}{q^2}\right) = 13.6\left(\frac{1}{q^2} - \frac{1}{p^2}\right)$ in eV.

The frequency $f_{photon} = 3.3 \times 10^{15}\left(\frac{1}{q^2} - \frac{1}{p^2}\right)$ in Hz.

As we have seen, some of these frequencies are in the visible region, meaning that we can see the process as an emission of visible light. (For hydrogen, those de-excitations with a final state $q = 2$ are in the visible region).

Example: If an electron drops from the 3rd to the 2nd orbital shell i.e. from $p = 3$ to $q = 2$ then a photon of frequency $f_{photon} = 3.3\times10^{15}\left(\frac{1}{2^2} - \frac{1}{3^2}\right)$ or $f_{photon} = 4.58\times10^{15}$ Hz corresponding to a wavelength of 656 nm, (red) as $c = f\lambda$.

Carrying out spectroscopy on other elements shows each to have a characteristic spectrum enabling the gaps between the electron energy levels to be calculated. This is an important analytical tool allowing us to recognise gases based on their absorption / emission spectra.



2.4 Multi-electron atoms

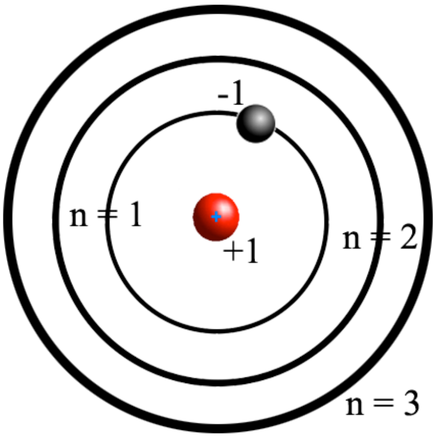
- The number of protons in the nucleus determines the name of the element.
- The number of neutrons in the nucleus is not strictly fixed and since the neutron is uncharged, this has no effect on the overall charge of the element. An **isotope** is defined as an atom of the same chemical element with a different number of neutrons but the same number of protons.
- The atomic number Z is the number of protons in the nucleus.
- The mass number A is the total number of protons and neutrons (called nucleons) in the nucleus.
- In an uncharged element, the number of electrons is equal to the number of protons. In an ionised element, the number of electrons can either be less (positive net charge) or more (negative net charge) than the number of protons.

Examples

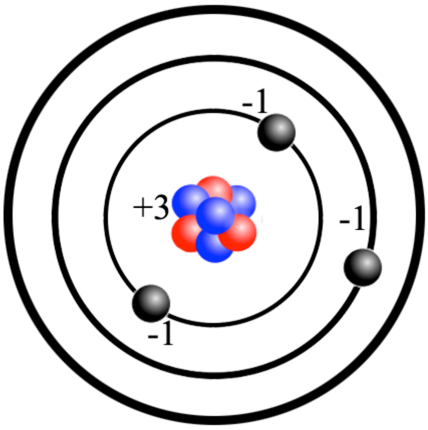
1. An uncharged hydrogen atom has a charge on the nucleus of $+1e$ which is balanced by the 1 electron in orbit around it. The same is true for all uncharged elements.
2. A helium atom ${}^{4}_{2}He$ has a mass number of 4 and an atomic number of 2. Therefore it contains 2 protons and since uncharged it must also contain 2 electrons. A mass number of 4 indicates that the nucleus contains 4 nucleons. Since we know it has 2 protons, the nucleus must contain 2 neutrons.
3. There exist many isotopes of carbon among them ${}^{13}_{6}C$ and ${}^{12}_{6}C$. Each contains 6 protons and 6 electrons. However since the mass number of each is different, they contain respectively 13 and 12 nucleons and therefore 7 and 6 neutrons.
4. An ion of copper ${}^{64}_{29}Cu^{2+}$ has a positive charge of +2 indicating that it contains 29 protons and 27 electrons. A mass number of 64 means that there are 64 nucleons in the nucleus of which 35 are neutrons.

Now the question is how, in a multi-electron atom, do these extra electrons fit into the Bohr model?

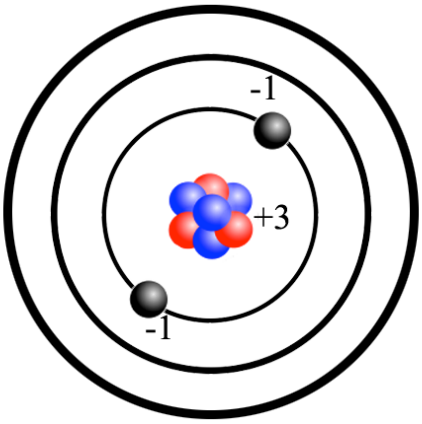
It turns out each 'shell' or orbital (corresponding to a particular n) can only contain up to a particular number of electrons. For example the orbital $n = 1$ can hold up to 2 electrons whereas the $n = 2$ orbital holds up to 8 electrons. In fact it can be shown that the nth orbital can accommodate up to $2n^2$ electrons.



Hydrogen atom ${}_1^1H$ (1p, 1e, 0n)



Lithium atom ${}_3^7Li$ (3p, 3e, 4n)



Lithium ion ${}_3^7Li^{1+}$ (3p, 2e, 4n)

All of these atoms or ions have their own energy level diagrams, with allowed energy levels just like the hydrogen atom, although always more complicated. If the Bohr energy level calculation is repeated for a nuclear charge of Ze rather than just e, the corresponding radii of the various orbitals is found to be

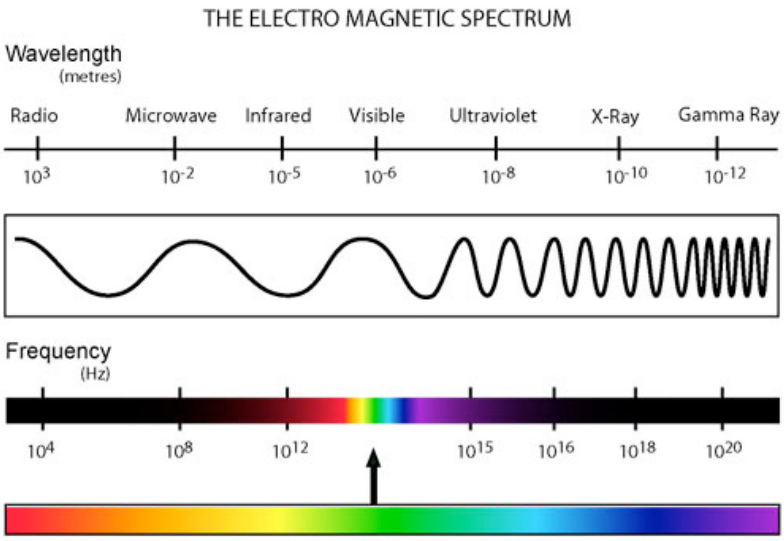
$$r_n = \frac{5.26\times10^{-11}\, n^2}{Z}$$ in metres and their energies to be $$E_n = \frac{-13.6\,Z^2}{n^2}$$ in eV.

Irrespective of the number of electrons in the atom, the innermost electrons closest to the nucleus are the most strongly bound and therefore the most difficult to remove. The outermost electrons normally only take a few eV or 10s of eVs to remove, and therefore take part in all forms of chemical reaction. Molybdenum, for example has $Z = 42$, so the innermost $n = 1$ orbital shell will require energies of approximately $13.6\times42^2 = 24$ keV to strip them from the atom. This is a great deal of energy.

2.5 X-rays generators

We have just seen that it is possible to excite the orbiting electrons in an atom to higher energy levels and that when they subsequently return to lower energy states, they emit photons. The higher the energy difference between the orbital shells, the higher the energy of the photon emitted.

Since $E_{photon} = h\,f$ we can see that the frequency of the emitted photon also increases with increasing energy. Looking at the full electromagnetic spectrum below, we see that frequencies above 10^{17} Hz correspond to X-rays and Gamma rays.



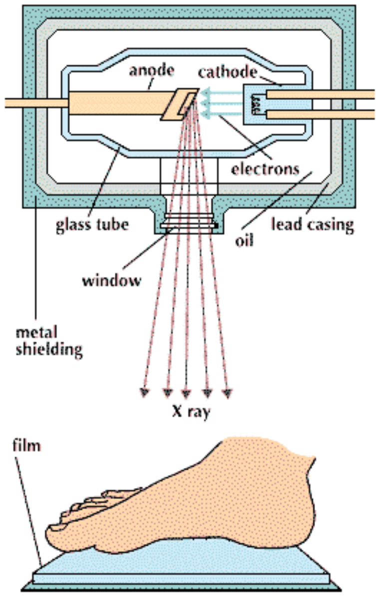
So what kind of energies must we supply to excite an orbiting electron which will subsequently decay to emit a 10^{19} Hz X-ray photon?

$E_{photon} = h\,f$ and so we must supply

$$E = \frac{6.626\times10^{-34}\times10^{19}}{1.6\times10^{-19}} = 41.4\ \text{keV}$$

What we need now is some kind of source of energy in the 10 -100 keV range to put them in this excited state.

One easy way to do this is to bombard the atoms with a high energy electron beam as shown in the diagram right. In the diagram electrons, generated by a hot wire filament, are accelerated from a cathode towards a metal target anode held at a high positive voltage to ensure that the incident electrons collide at high velocity.



In reality approximately 90% of the kinetic energy of the incident electrons is converted to heat energy. The remaining energy excites the orbital electrons of the target atoms which subsequently de-excite emitting photons. Obviously the range of photon energies produced is determined by the number of allowed transitions between various orbital shells. By careful selection of the target atoms and the energy of the incident electrons, a beam of mono-energetic X-rays or even Gamma-rays can be produced. X-ray generators produce a strange characteristic spectrum which is the result of two different processes which are explained below. The resultant final spectrum is shown on the next page, whilst the origin of the characteristic shape is explained below.

The final spectrum, shown on the following page, is made up of two physical processes:

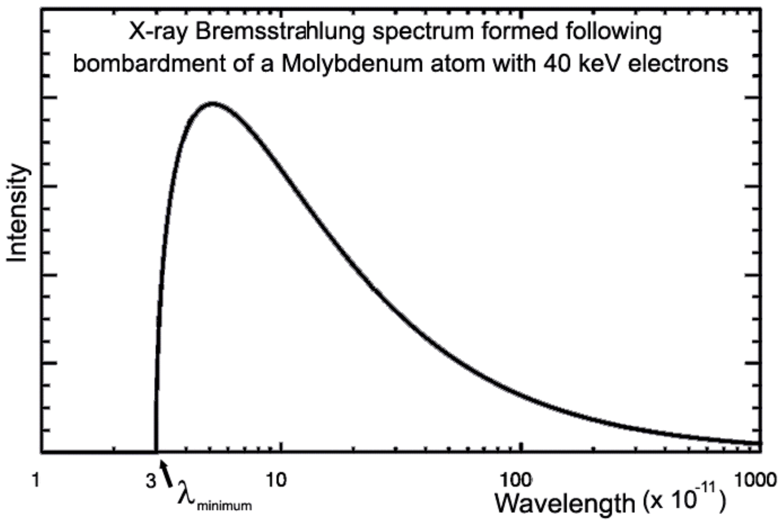
1. The incident electrons lose energy in inelastic collisions with the target atoms. The lost kinetic energy is emitted as X-ray photons; the incident electrons are slowed down in this process and therefore the X-ray are also called Bremsstrahlung radiation (meaning braking radiation in German). Incident electrons can lose any fraction of their energy in a collision meaning the X-ray photons emitted cover a continuous range of frequencies resulting in a continuous spectrum as shown in the figure.

   The spectrum also shows a well defined minimum wavelength (maximum frequency). The shape of the continuous Bremsstrahlung spectrum depends only upon the energy of the electrons and not upon the nature of the target. The same continuous spectrum is therefore seen for all metals. There will always be a clearly defined minimum energy corresponding to the largest energy photon that can be emitted. This refers to a collision in which an incident electron converts all its energy into a photon of identical energy.

   Do you see why the minimum X-ray wavelength produced by 40 keV electrons is $\lambda_m = 3\times10^{-11}\,m$?

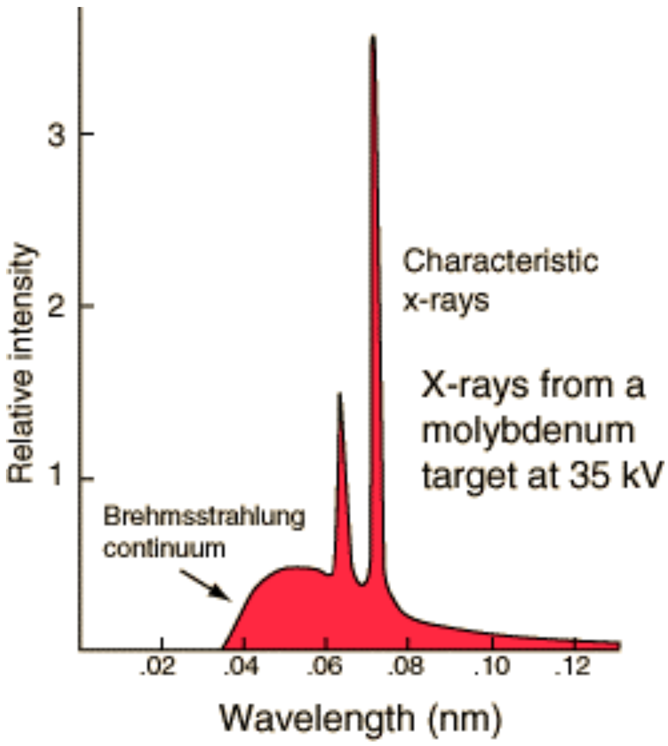
   $$E_{photon} = hf = \frac{hc}{\lambda} \quad \text{so} \quad \lambda_m = \frac{hc}{E_{photon}}$$

   $$\lambda_m = \frac{6.626\times10^{-34}\times3\times10^{8}}{40000\times1.6\times10^{-19}} = 3.1\times10^{-11}\,m$$



2. (This is the standard photoelectric effect that we have already met). The incident electrons lose energy in collisions with the target atoms exciting the orbiting electrons, which on returning to the ground state emit photons of well defined energies producing X-ray line spectra.

If the complete X-ray spectrum created by 40 keV incident electrons is generated, it will show both the broad Bremsstrahlung spectrum (1) and the emission lines (2) superimposed on it. The figure below shows this. The wavelengths defined by the line spectra in this figure depends only on the nature of the target atoms, i.e. the X-ray emission lines are specific to that atom.



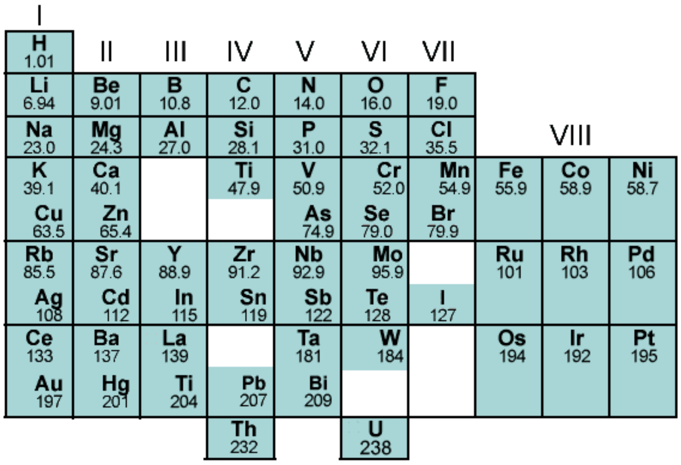
2.6 Applications of the photoelectric effect

- Medical and other imaging: it's well known that short wavelength waves (e.g. X-rays) pass easily through many solids; nevertheless, there is some absorption of them and different solids absorb different amounts. Hence bone looks darker than flesh in an X-ray image.
- Material analysis: the lines are characteristic of the element, so we can find out what an unknown sample of material contains by firing an electron beam at it and analysing the emitted X-ray wavelengths.

Chapter 3: The nucleus

3.1 The neutron, isotopes, and relative atomic mass

Because of Rutherford we know that the nucleus is about 10^5 times smaller than the atom, contains all the positive charge, and all the mass. The only particle we know of that is positively charged (charge = *e*) and massive compared with the electron is the proton (mass 1.6726×10^{-27} kg or 1837 × the electron mass). For a long time therefore, this became the accepted model.



At the same time as physicists were investigating the structure of the atom, chemists were investigating the chemistry of the elements. Avogadro's law states that equal volumes of gases, at the same temperature and pressure, contain the same number of atoms. From this hypothesis by comparing the weights of equal volumes of different gases is was possible to ascertain the relative weights of the atoms of each gas. This concept was gradually developed until in 1869 Mendeleev published the first periodic table listing both the relative atomic mass and atomic number of all known elements. The relative atomic mass is defined as the mass of an atom compared with the mass of an atom of carbon-12 taken to be exactly 12. (Relative atomic masses have no units because they are a ratio of two variables with the same units. We'll look at the difference between mass number and relative atomic mass in the next section).

However, physicists immediately saw a big problem. The relative atomic mass is not equal to Z as it would be if there were only protons present, but nearer to $2 \times Z$ (very roughly)!!! Clearly there must be something else in the atom which is both massive and neutral. This particle was discovered in 1932 by Chadwick and was called the neutron. Its mass has been calculated as very close to, but slightly greater than (more on this later) the proton. As we have already seen the total number of particles (collectively called nucleons) in the nucleus is referred to as the mass number. It is always an integer, denoted by the letter A. So the number of neutrons present is $A - Z$.

Why are the relative atomic masses calculated by chemists **not** integer values? How can we have a non whole number of nucleons!!? The answer of course is that the chemical methods used are unable to separate the various isotopes of each element and therefore give an average value for the number of nucleons present.

Example: Chlorine exists as two isotopes: chlorine-35 and chlorine-37. In a naturally occurring sample of 100 chlorine atoms, we find that 75 atoms are chlorine-35 and the other 25 atoms are chlorine-37. The total mass number for all 100 atoms will be $(75\times35)+(25\times37)=3550$. *The relative atomic mass is therefore* $\frac{3550}{100}=35.5$ *whereas the mass number of an individual chlorine atom can only be either 35 or 37 depending on the atom.*

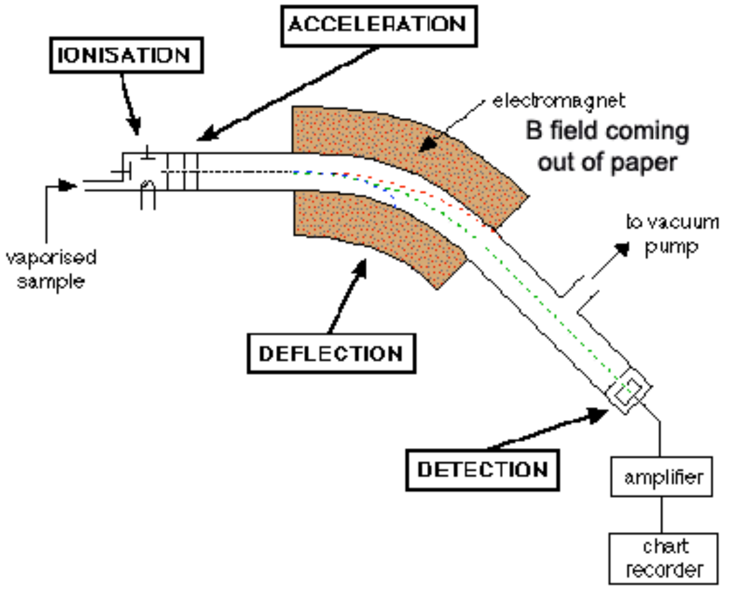
Remember….

- The atomic number Z determines the number of protons in the nucleus and the name of the element.
- An isotope is defined as an atom of the same chemical element with a different number of neutrons but the same number of protons.
- The mass number A is the total number of protons and neutrons (called nucleons) in the nucleus.
- The relative atomic mass is the average mass of all isotopes in a random sample of the element, taking into account the proportion of each isotope present.

3.2 Measuring mass – the mass spectrometer

So how do physicists separate isotopes if they cannot be identified chemically? The answer is that we must again make use of the techniques used to define the charge / mass ratio back in section 1.3. The detector we use to do this is called a mass spectrometer. It consists of a straight tube connected to a bent section surrounded by an intense magnetic field with a detector at the end. The device is designed around the idea that although different isotopes have the same numbers of protons and electrons, they contain slightly different numbers of neutrons, meaning that they have different masses. Because of this mass difference they will be deflected by differing amounts as they pass through a magnetic field.

Obviously atoms have a net charge of zero and so will be unaffected by electric and magnetic fields. Atoms must first be turned into ions by knocking one or more orbital electrons out of each atom. This is done by bombarding them with a high energy stream of electrons, which produces positive ions. This process is carried out even for things which you would normally expect to form negative ions (e.g. chlorine), or never form ions at all (e.g. argon). Mass spectrometers always work with positive ions. Most of the ions created will have a charge of 1+ simply because it is increasingly more difficult to remove orbital electrons once the atom has a net positive charge.



A voltage V then accelerates the ions into the tube with a velocity v which can be calculated using the equation derived earlier in section 1.6 as : $v = \sqrt{\dfrac{2qV}{m}}$ .....(3.1)

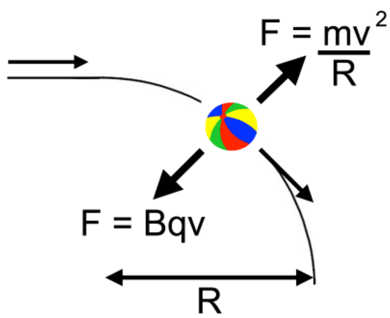
where q is the charge on the ion, V is the accelerating voltage, and m is the mass of the ion.

Passing towards the curved section at constant velocity, they enter a magnetic field B arranged at right angles to the plane containing the tube and coming out of the paper. At this point, they are deflected by the strong magnetic field according to their charge and mass. The lighter they are or the more highly charged they are, the more they are deflected. The force they experience, using Fleming's left hand rule, directed towards the centre of the curve given by :

$$F_{magnetic} = B\,q\,v \qquad .....(3.2)$$

As they start to curve they also feel a centrifugal force directed in the opposite direction given by :

$$F_{centrifugal} = \frac{m v^2}{R} \qquad .....(3.3)$$



Travelling in a circle of constant radius R, these two forces must balance and so we can say :

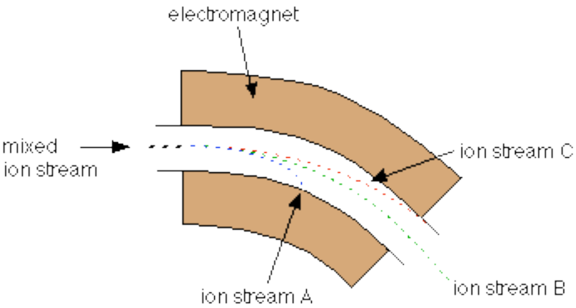
$$B\,q\,v = \frac{m v^2}{R} \qquad .....(3.4)$$

So $\dfrac{q}{m} = \dfrac{v}{B\,R}$ and then substituting from equation (3.1) we say $\dfrac{q}{m} = \dfrac{1}{B\,R}\left(\dfrac{2qV}{m}\right)^{\frac{1}{2}}$

This can be written as $\dfrac{q}{m} = \dfrac{\sqrt{2V}}{B\,R}\left(\dfrac{q}{m}\right)^{\frac{1}{2}}$ and so $\left(\dfrac{q}{m}\right)^{\frac{1}{2}} = \dfrac{\sqrt{2V}}{B\,R}$

We can therefore state that the charge / mass ratio is $\boxed{\left(\dfrac{q}{m}\right) = \dfrac{2V}{B^2 R^2}}$

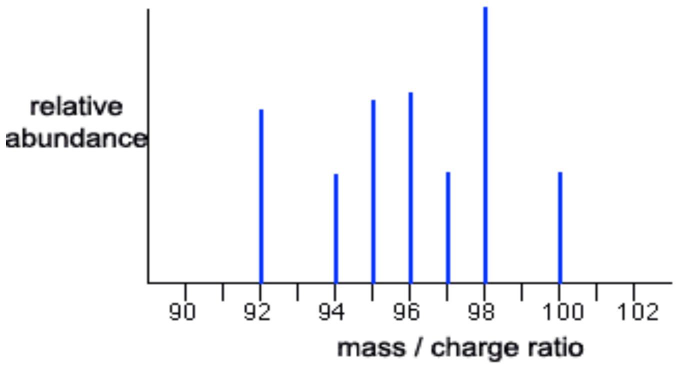
In the diagram below, ion stream A is most deflected - it contains ions with the greatest charge / mass ratio. Ion stream C is the least deflected - it contains ions with the smallest charge / mass ratio. Assuming all ions have a charge of 1+, stream A has the lightest ions, stream B the next lightest and stream C the heaviest.



How might the other ions be detected - those in streams A and C which have been lost in the machine? Remember that stream A was most deflected - it has the greatest value of charge / mass ratio. To bring them on to the detector, you would need to deflect them less - by using a smaller magnetic field (a smaller sideways force). To bring those in stream C, with a smaller charge / mass value, on to the detector you would have to deflect them more by using a larger magnetic field.

Finally the ions reach a detector at which point their exact position is recorded. This is then used to calculate the radius of deflection which provides a measure of the charge / mass ratio. A typical output from a mass spectrometer looks like this one for molybdenum. To avoid tiny decimals we usually plot mass / charge ratio rather than charge / mass ratio. This has the added advantage that for ions with a 1+ charge, the ratio is also equal to the mass number of the atom.

The vertical scale is related to the number of ions arriving at the detector. As you will see from the diagram, the commonest ion has a mass / charge ratio of 98. Other ions have mass / charge ratios of 92, 94, 95, 96, 97 and 100. That means that molybdenum consists of 7 different isotopes. The relative atomic mass of molybdenum can then be calculated with respective to the relative abundances as previously done for chlorine.



Incidentally, if there were also 2+ ions present, you would know because every one of the lines in the diagram above would have another line at exactly half its mass / charge value (because, for example, 98/2 = 49). Those lines would be much less tall than the 1+ ion lines because the chances of forming 2+ ions are much less than forming 1+ ions.

All elements have isotopes. We write them as $^{A}_{Z}X$ where X is the element symbol. Neon has 3 isotopes which are written $^{22}_{10}Ne$, $^{21}_{10}Ne$, and $^{20}_{10}Ne$. Hydrogen has three: $^{1}_{1}H$, $^{2}_{1}H$, and $^{3}_{1}H$. The fact that many relative atomic masses are nearly integer is because one isotope is often more common than the others.

3.3 What holds the nucleus together?

Although neutrons have no charge, protons have a charge of 1+ and so we would expect them to repel one another in the nuclei. So what attractive force keeps them together? It's clearly not an electromagnetic force, because protons have the same sign charge so *repel* each other. It's also not gravity, because the gravitational force between nucleons is incredibly small. We call this force the strong force.

The strong force is one of the four fundamental forces, along with gravitation, the electromagnetic force and the weak interaction. The word *strong* is used since the strong interaction is the most powerful of the four fundamental forces; its typical strength is 100 times that of the electromagnetic force, some 10^{13} times as great as that of the weak force, and about 10^{38} times that of gravitation.

It pulls together all nucleons irrespective of whether they are protons or neutrons. Because it is a strong but short-ranged force ($<5\times10^{-15}$ m) whereas the electromagnetic force is long ranged in comparison, we find that for separations greater than atomic spacings, the familiar electromagnetic repulsion between identical polarities dominates.

3.4 Binding energy

According to special relativity, a mass *m* is equivalent to an amount of energy *E* where:

$$E = mc^2$$

where *c* is the speed of light. It follows that whenever a reaction results in the release of energy, there must have been as associated decrease in the mass of the products. For example, when 1kg of ${}^{235}_{92}U$ undergoes fission (see later) the energy released is approximately 8×10^{13} joules and therefore there must have been a decease in mass of $\frac{8\times10^{13}}{(3\times10^8)^2} = 9\times10^{-4}$ kg. This is a significant fraction of the initial mass and can easily be measured. This is not generally true, for example when 1 kg of petrol is burned the energy released corresponds to a mere 5.5×10^{-10} kg mass difference.

We now need a convenient unit to allow us to calculate this mass difference easily for reactions. We have already said that the relative atomic mass is defined as:

$$RAM = \frac{\text{the average mass of the atom}}{\text{One twelfth the mass of a } {}^{12}_{6}C \text{ atom}}$$

It follows that the *RAM* of ${}^{12}_{6}C$ is exactly 12, whereas that of hydrogen is 1.008 and oxygen is 15.995.

The atomic mass unit is defined such that the mass of a ${}^{12}_{6}C$ atom is 12u exactly. It can be shown that the atomic mass unit therefore has a mass of:

$$1\text{u} = 1.66\times10^{-27}\text{ kg}$$

And this can be converted into an energy of $1.66\times10^{-27}\times(3\times10^8)^2 = 1.49\times10^{-10}$ joules. If this energy is written in electron-volts we can state:

$$1\text{u} = 931.5\text{ MeV}$$

It can be shown that a proton has a mass of 1.00728u, a neutron has a mass of 1.00867u, and an electron has a mass of 0.00055u.

The mass of a nucleus is always less than the total mass of its constituent protons and neutrons, the difference in mass called the mass defect.

$$\text{Mass defect} = (\text{Mass of separate nucleons and electrons}) - (\text{Mass of atom})$$

The reduction in mass arises because the act of combining the nucleons to form the nucleus causes some of their mass to be released as energy in the form of gamma rays. This has the effect of binding the nucleons together since any attempt to separate them would require energy to be supplied. This energy (the added energy needed to take a nucleus apart into its constituent protons and neutrons, or conversely the energy released when all the nucleons come together) is called the binding energy of the nucleus and it follows from above that this can be written as:

$$\text{Binding energy} = \text{Mass defect} \times c^2$$

Or from the above:

$$\text{Binding energy (MeV)} = 931.5 \times \text{Mass defect (u)}$$

Example: Consider the helium atom ${}^{4}_{2}He$. It consists of 2 protons, 2 neutrons, and 2 electrons. If the mass of a helium atom is 4.0026u what is its binding energy?

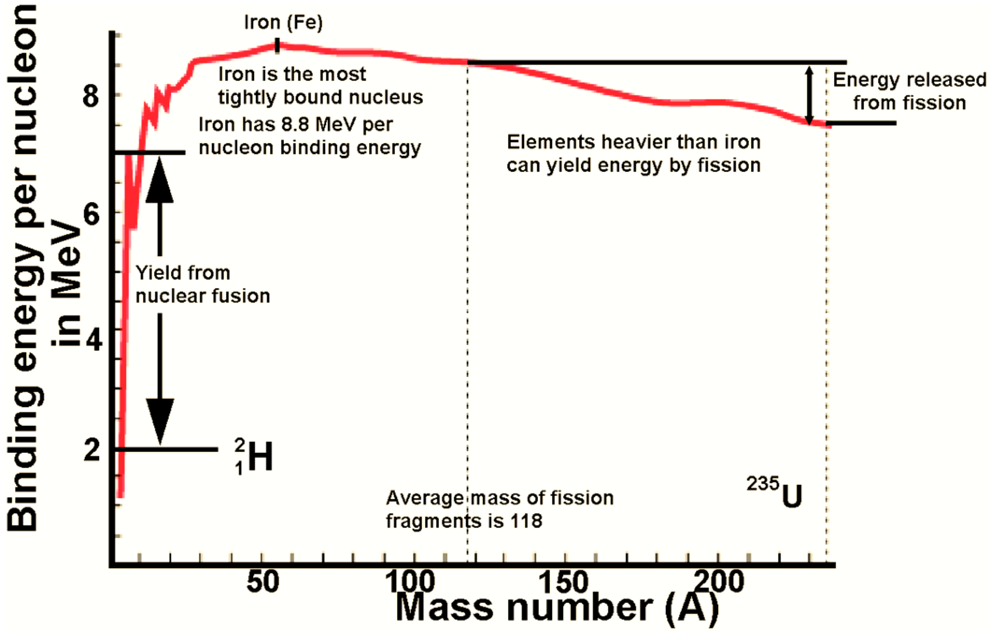
The total mass of all constituents is $(2\times1.00728\text{u}) + (2\times1.00867\text{u}) + (2\times0.00055\text{u}) = 4.033\text{u}$.

The mass defect in amu is therefore: $\text{Mass defect} = 4.033\text{u} - 4.0026\text{u} = 0.0304\text{u}$

This corresponds to a binding energy (MeV) of $931.5 \times \text{Mass defect (u)} = 931.5 \times 0.0304\text{u} = 28.3\text{ MeV}$.

3.5 Nuclear stability and mass number

A useful measure of the stability of the nucleus is its binding energy per nucleon (i.e. the total binding energy divided by the mass number) since this represents the average energy needed to be supplied to remove a nucleon. The figure shows the way that this quantity varies with mass number. Atoms with mass numbers from approximately 30 to 80 have the largest binding energy per nucleon and are therefore some of the most stable nuclei. $^{56}_{26}Fe$ has a value of 8.8 MeV and is the most stable nuclei.



- The rising of the binding energy curve at low mass numbers, tells us that energy will be released if two nuclides of small mass number combine to form a single middle-mass nuclide. This process is called nuclear fusion.
- The eventual dropping of the binding energy curve at high mass numbers tells us on the other hand, that nucleons are more tightly bound when they are assembled into two middle-mass nuclides rather than into a single high-mass nuclide. In other words, energy can be released by the nuclear fission, or splitting, of a single massive nucleus into two smaller fragments. (We will cover fission and fusion in the next chapter).

The rapid rise of the binding energy per nucleon and therefore the mass defect for the first few elements suggests that a fusion reaction would be extremely exothermic. At the time, knowing that stars are composed mostly of hydrogen and helium, scientists speculated that energy from starlight was likely the result of hydrogen fusing to helium. After World War II, the United States developed a hydrogen bomb powered by fusion which was found to be much more powerful than the atomic fission bomb.

NB. The binding energy per nucleon is always a positive value meaning that the mass of the nucleus is always smaller than the mass of its component parts. The fission and fusion processes release energy because of the difference between the binding energies of different nuclei which can be either positive or negative depending on the mass number of the original nuclei and the type of reaction.

Let's look at an example: Will the fission of helium-4 into helium-3 and a neutron emit energy or require energy to be added? (mass of helium-4 is 4.001505u, helium-3 is 3.014931u, and a neutron is 1.00867u).

$$^4_2He \rightarrow \, ^3_2He + \, ^1_0\text{n}$$

So mass on the left is 4.001505u whilst total mass on right is 4.023601u. The mass defect is 0.022096u.

The fact that this is a positive value in the direction of the reaction indicates that energy must be supplied for this reaction to occur. Energy required $= 931.5 \times \text{mass defect (u)} = 931.5 \times 0.022096\text{u} = 20.6\,\text{MeV}$.

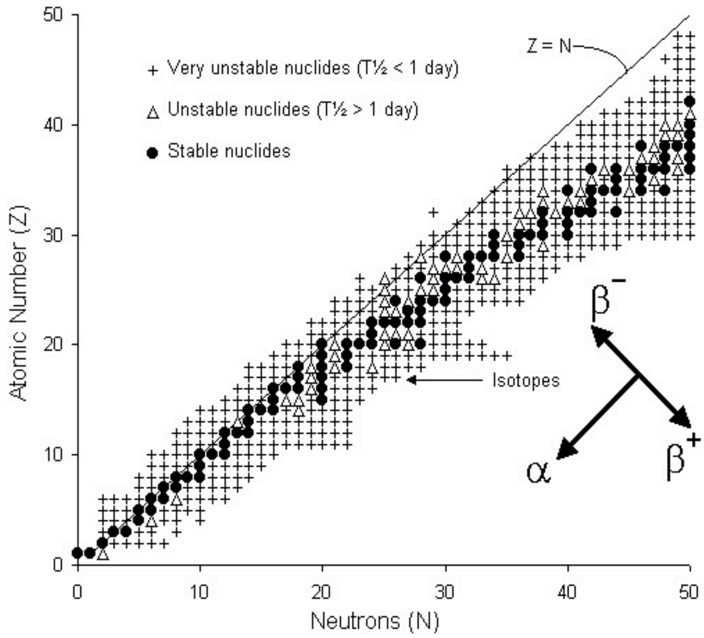
Energy is emitted in any reaction for which the total mass difference in that direction is negative.

Chapter 4: Radioactivity

4.1 Radioactivity and stability

The plot of binding energy against mass number only lists the stable nuclei. However, most elements have many isotopes, most of which are unstable, usually decaying on timescales from microseconds to billions of years into another element frequently with a change in the number of the nucleons.

Radioactive decay only takes place if the energy of the daughter element is lowered relative to the energy of the parent, the excess energy given to the emitted particle. This implies that the total mass on the left side of the equation must be greater than on the right. We will look at the various forms of radioactive decay shortly.



If we plot the number of neutrons against the number of protons for stable nuclei only, it is evident that heavy elements have more neutrons than protons. Small nuclei require approximately equal numbers of neutrons and protons for stability. When we look at larger and larger nuclides having more and more positive charge within the nucleus, we see that a higher percentage of neutrons are needed to shield the repulsive positive charges from one another. Finally, there comes a point where no amount of neutrons can make the nuclide stable. There are just too many positive charges. There is no element with an atomic number $Z > 82$ (lead) which has any stable isotopes. ${}^{208}_{82}\text{Pb}$ is the heaviest stable isotope.

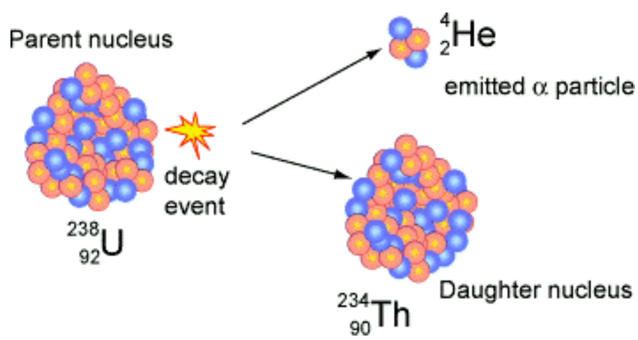
Based on its position on the plot, it is possible to predict the most likely form of radioactive decay for an isotope in order for it to become more stable, as shown on the figure. We will now describe the four types of radioactive decay.

4.2 α decay – loss of a He nucleus

The α particle is simply the nucleus of the helium atom $\left({}^{4}_{2}He\right)$ in that it is made up of 2 protons and 2 neutrons, and therefore has a charge of 2+. An unstable element emitting an α particle thereby produces a new nucleus with mass number A - 4 and atomic number Z – 2. This decay mechanism is most common in *heavy* elements as can be seen from the plot above.

An example is: ${}^{238}_{92}\text{U} \rightarrow {}^{234}_{90}\text{Th} + {}^{4}_{2}\text{He}$ (half-life is 4.47×10^{9} years).

Another example is: ${}^{218}_{86}\text{Rn} \rightarrow {}^{214}_{84}\text{Po} + {}^{4}_{2}\text{He}$ (half-life is 35 ms).



For each example the proton (Z) and nucleon numbers (A) add up on each side e.g. 92 = 90 + 2 and 238 = 234 + 4. Typical α particle energies are around 5 MeV.

4.3 β decay – neutrons turn to protons and vice versa

4.3.1 Particles and anti-particles

The particles in the nucleus (protons and neutrons) and orbiting around it (electrons) are part of a larger collection called fermions. It turns out that for each of these particles there is an anti-particle of the same mass. The difference between particle and anti-particle is most obvious when they are charged, because although the magnitude of the charge is the same on particle and anti-particle, the sign is different. For example, the anti-electron (known more commonly as the positron has charge $+e$ in contrast to the electron's $-e$. We write them as e^- (electron) and e^+ (positron) or in the context of radioactivity as β^- and β^+.

4.3.2 Neutrino

This is another particle that is involved in the β decay processes. Though neutral and nearly mass-less, the neutrino also has an anti-particle called the anti-neutrino. Millions of anti-neutrinos, generated in the Sun pass through us every second without interacting. We write them as ν and $\bar{\nu}$, the bar used to show which is the anti-particle.

4.3.3 β^- decay

β^- decay involves the emission of an electron from the **nucleus** of the atom. But where did this electron come from?? What has happened is that one of the neutrons in the nucleus has transformed into a proton, with the emission of an electron and an anti-neutrino. (In order to more easily balance the radioactive decay process, we usually write an electron as ${}^{0}_{-1}e$).

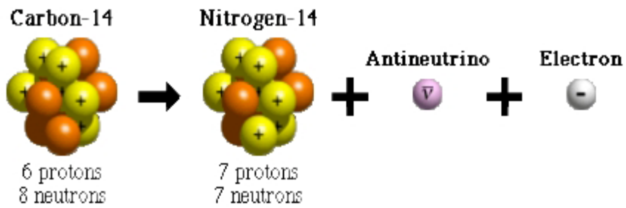
$${}^{1}_{0}n \rightarrow {}^{1}_{1}p + {}^{0}_{-1}e + \bar{\upsilon}$$

Note that charge, A, and Z are conserved (same on the left and the right of the equation). As with all radioactive decay there is more mass on the left (1.00867u) than on the right (1.00728u + 0.00055u = 1.00783u) and so the energy of the daughter element is lower than that of the parent, the spare energy given to the ejected β^- particle and anti-neutrino. But how much energy is this?

The excess in atomic mass units is 8.4×10^{-4}u. Since 1 a.m.u. corresponds to 931.5 MeV then this mass corresponds to an energy of $8.4\times10^{-4}\times931.5$ = 782 keV which is shared between the electron and the anti-neutrino. (That is how we know the anti-neutrino is there: the electron energy is not a single value but covers a range with a well defined maximum where it has maximum energy and the $\bar{\nu}$ has none).

The overall effect of β^- decay on the nucleus is therefore to swap a neutron to a proton. The mass number is the sum of protons and neutrons and this is therefore unchanged. The creation of a proton however causes the atomic number Z to increase to Z + 1.

An example is:	${}^{14}_{6}C \rightarrow {}^{14}_{7}N + e^- + \bar{\nu}$ (half-life is 5730yr).
Another example is:	${}^{239}_{92}U \rightarrow {}^{239}_{93}Np + e^- + \bar{\nu}$ (half-life is 23.54 min).



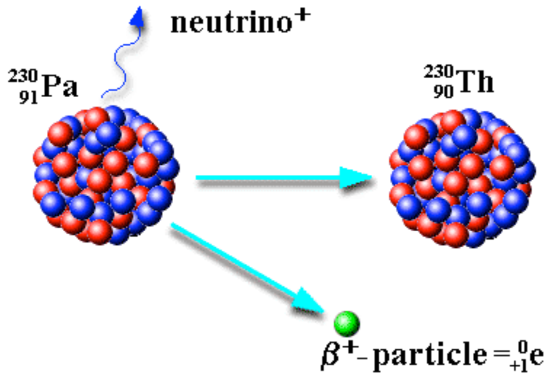
4.3.4 β$^+$ decay

There is also a very closely related process called β$^+$ decay in which a proton in the parent nucleus transforms into a neutron, a positron, and a neutrino. On its own the proton does not decay (the reaction requiring energy), but inside the nucleus that energy may be available allowing the following reaction to occur:

$$ {}^{1}_{1}\mathrm{p} \rightarrow {}^{1}_{0}\mathrm{n} + {}^{0}_{1}\mathrm{e} + \upsilon $$

Again in order to more easily balance the radioactive decay process, we usually write a positron as ${}^{0}_{1}\mathrm{e}$.

An example is:	${}^{14}_{8}\mathrm{O} \rightarrow {}^{14}_{7}\mathrm{N} + {}^{0}_{1}\mathrm{e} + \upsilon$ (half-life is 70.6 seconds).
Another example is:	${}^{230}_{91}\mathrm{Pa} \rightarrow {}^{230}_{90}\mathrm{Th} + {}^{0}_{1}\mathrm{e} + \upsilon$ (half-life is 17.4 days).



In this process a proton in the parent nucleus transforms into a neutron so decreasing the atomic number from Z to Z – 1. Again the mass number does not change. Another name for this form of radioactivity is positron emission, and reactions like those above are used in PET scanners.

4.4 γ decay – nuclear reorganisation

This process, unlike those listed above, involves no change in the numbers of protons or neutrons in the nucleus. γ decay occurs when a nucleus finds itself in an excited state after having undergone either α or β$^-$ decay, the new arrangement of nucleons such that they may hold extra energy in the form of, for example, electrical potential. The nucleus may then be able to reorganise its nucleons into a lower energy state accompanied by the emission of a photon.

NB. This is different to the emission of a photon from the hydrogen atom covered earlier when discussing the photoelectric effect for which an electron could be put into an excited orbital state, and returned to the ground state with emission of a photon.

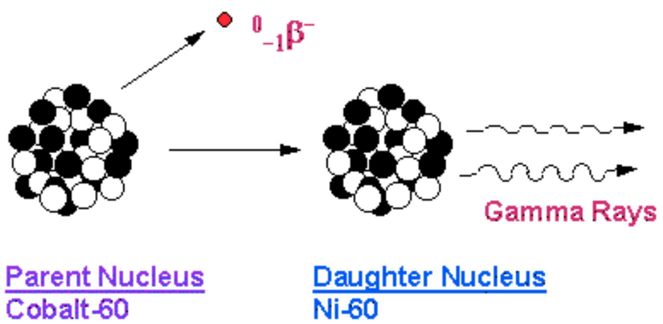
The excited nuclear energies are so much bigger than energies associated with excited electron orbital states. Hence γ-rays are emitted following reorganisation of the nucleus, with very short wavelengths as $E_{photon} = hf = \frac{ch}{\lambda}$. Typical energies are 10 keV to 10 MeV. Like the line emission spectra of atomic transitions the nuclear reorganisations also produce well defined photon energies.

Below is an example of γ-ray emission from an excited nickel nucleus following a β$^-$ decay from cobalt, the * indicating an excited nucleus.

$$ {}^{60}_{27}Co \rightarrow {}^{60}_{28}Ni^{*} + e^{-} + \overline{\nu} $$

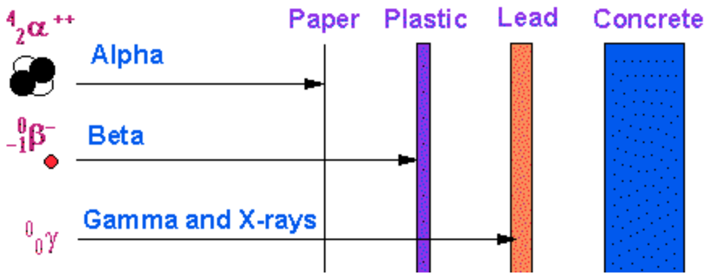
followed by

$$ {}^{60}_{28}Ni^{*} \rightarrow {}^{60}_{28}Ni + \gamma $$



4.5 Penetration distance

Alpha particles can usually be stopped by paper. Radioisotopes emitting alpha particles are usually not hazardous outside the body, but they can cause damage if ingested.



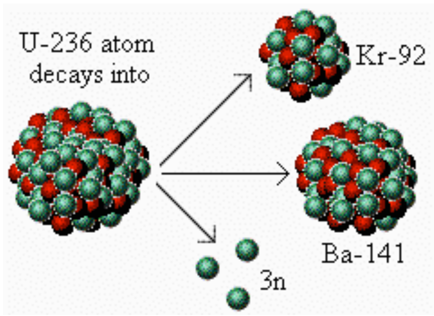
Betas can pass through a hand, but are usually stopped by a modest barrier such as plastic. Beta particles are more hazardous if inhaled or ingested.

Gammas are very penetrating and can pass through thick barriers. Several feet of concrete would be needed to stop some of the more energetic gammas.

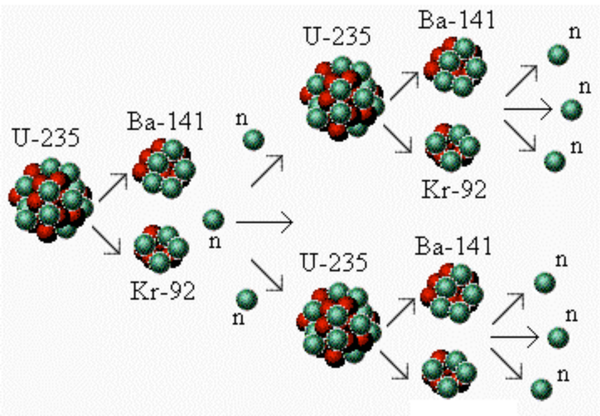
4.6 Fission

In section 3.5 we saw how energy can be released by the nuclear fission, or splitting, of a single massive nucleus into two smaller fragments. This is the basis of nuclear power generation. Fission can be spontaneous, but only for very heavy nuclei such as uranium-238, and even then, the rate is very low. Fission of a heavy nucleus such as uranium-235 $\left({}^{235}_{92}\mathrm{U}\right)$ can alternatively be induced for example by the absorption of a neutron, producing krypton-92, barium-141, three neutrons, and 174 MeV of energy:

$$ {}^{235}_{92}\mathrm{U} + {}^{1}_{0}\mathrm{n} \rightarrow {}^{92}_{36}\mathrm{Kr} + {}^{141}_{56}\mathrm{Ba} + 3\,{}^{1}_{0}\mathrm{n} + 174\,\mathrm{MeV} $$



Following absorption of the initial neutron, the uranium-236 nucleus splits into two daughter products and releases three additional neutrons. These neutrons quickly cause the fission of other uranium-235 atoms, thereby releasing additional neutrons and initiating a self-sustaining series of nuclear fissions, or a chain reaction shown below, which results in continuous release of nuclear energy.



For the same mass, a single nuclear fission reaction releases 10 million times as much energy as is released in the burning of fossil fuels. The fission of 1 kg of uranium-235 releases 18.7 million kilowatt-hours of energy in the form of heat. Put another way, if you currently use a tank of petrol each week but could use the energy provided by one tank of uranium-235 fission instead, you wouldn't need to re-fill your car for over 19,000 years!

If we want to know exactly how much energy is released we calculate the mass defect as before. The masses are given in the table below.

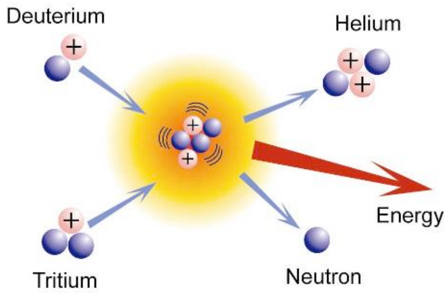
Nucleus type	Uranium-235	Barium-141	Krypton-92	Neutron
Mass a.m.u.	235.04393u	140.914411u	91.926156u	1.00867u

The mass difference between the left and right sides of the reaction is 0.186033u. Since 1 a.m.u. corresponds to 931.5 MeV then this corresponds to 174 MeV.

4.7 Fusion

In section 3.5, the rising of the binding energy curve at low mass numbers, tells us that energy will be released if two nuclides of small mass number combine to form a single middle-mass nuclide. This process is called nuclear fusion.

An example below shows the fusion of a deuterium and a tritium nuclei producing a helium isotope, a neutron, and 17.6 MeV. (Deuterium and tritium are isotopes of hydrogen in that they contain extra neutrons in the nuclei).



$$ {}^{2}_{1}\mathrm{H} + {}^{3}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + {}^{1}_{0}\mathrm{n} + 17.6\,\mathrm{MeV} $$

In order to initiate fusion we must raise the temperature of the nuclei so that the particles have enough energy - due to their thermal motion alone - to overcome their mutual electrostatic repulsion. This process is known as thermonuclear fusion. Calculations show that these temperatures need to be close to the sun's temperature of 1.5 X 10^7K. In fact reactions of this type provide the Sun’s energy. To date fusion on Earth has only been achieved using a hydrogen bomb, in which the intense temperature required for the fusion reaction is provided by the explosion of a fission bomb.

As can be seen, the energy released by the fusion of two nuclei is very much less than that which results from fission of a uranium nucleus. If we want to know exactly how much energy is released we calculate the mass defect as before. The masses are given in the table below.

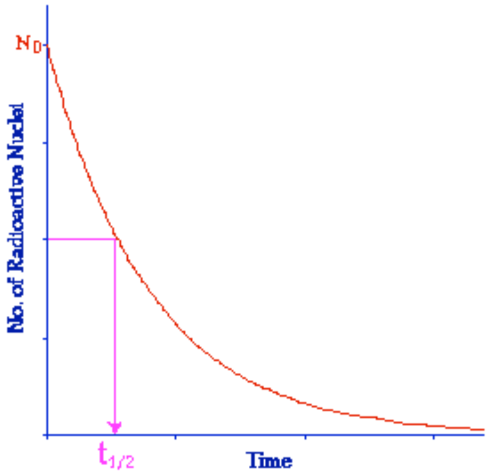
Nucleus type	Deuterium	Tritium	Helium	Neutron
Mass a.m.u.	2.0141018u	3.016049u	4.002603u	1.00867u

The mass difference between the left and right sides of the reaction is 5.0301508u – 5.011268u = 0.0188828u. Since 1 a.m.u. corresponds to 931.5 MeV then this corresponds to 17.61 MeV.

4.8 Exponential decay and half-life

Radioactive decay is a random process, although this doesn’t mean that it is necessarily slow. If we start with a certain number of parent atoms n_0 and note the number $n(t)$ remaining after time *t*, then we find that the resulting plot is an exponential decay which can be fitted by the equation:

$$ \boxed{n(t) = n_0 \exp(-\lambda t)} $$



NB. Although *n(t)* typically refers to the number of parent atoms present at time *t*, it can also represent the decay rate of the parent (i.e. number of emitted alphas per second for example). So long as n_0 is given in the same units, this is fine.

The number of parent nuclei or decay rate therefore drops exponentially with time. The constant λ (units s^{-1}) depends on the isotope and the kind of radioactive decay. The bigger λ is the faster the decay. Let’s look at some features.

i) We can easily work out the number of parent atoms *n(t)* at any time *t* if we know the original value n_0 at the start and the constant λ.

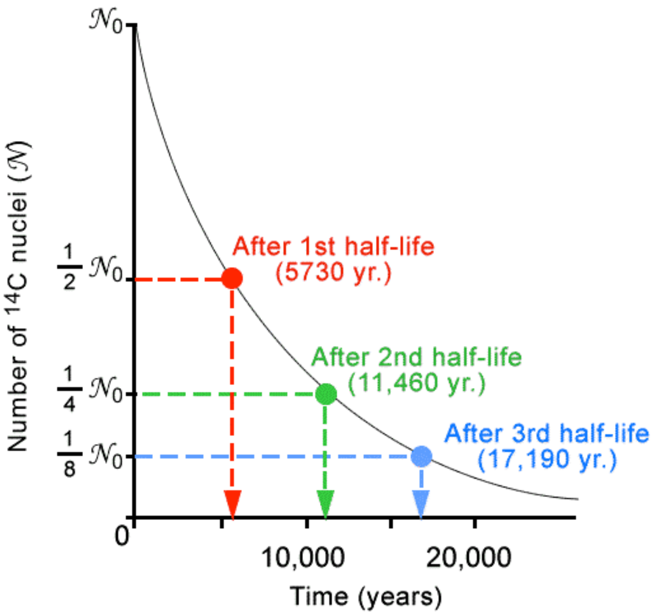
Example: Suppose a box contains 2000 particles at $t = 0$ and $\lambda = 4.5\times10^{-4}\ s^{-1}$. Then after an hour the number of parent atoms in the box will be $2000\times\exp(-4.5\times10^{-4}\times3600) = 396$ particles.

ii) The rate at which isotopes decay is usually illustrated by quoting the time taken for half of the original (parent) atoms to decay. This is termed the half-life and is the time taken for the number of parent atoms to decrease by a factor of 2 as shown in the figure.

Since we know that $n(t) = n_0 \exp(-\lambda t)$ then we can also say that $\frac{n}{n_0} = \frac{1}{2} = \exp(-\lambda t_{half})$. Therefore taking logs of both sides we find $\ln\left(\frac{1}{2}\right) = -\lambda t_{half}$ and so we can write:

$$t_{half} = \frac{0.693}{\lambda}$$

The constant λ is therefore not the same as the half-life t_{half} but is related to it.



Example: If the half-life is 1500 years, then $\lambda = \frac{0.693}{t_{half}} = \frac{0.693}{1500 \times 365 \times 24 \times 3600} = 1.46\times10^{-11}\ s^{-1}$.

Let's now look at the way we derived the expression $n(t) = n_0 \exp(-\lambda t)$ for radioactive decay. (You don't need to remember this for the exam, but you should make sure you understand the steps).

Why does the equation have this form? Think about the number *n(t)* of parent atoms in a sample at any time *t*. The decay process is a totally random one and we are unable to say whether or not an atom will decay at a certain time. However what we can say is that the rate of change in the number of parent atoms present is directly proportional to the number of atoms present, or in calculus terms: $\frac{dn}{dt} = -\lambda n$ where λ is the constant of proportionality. To solve this equation using calculus we move all the terms in n to one side and all the terms in t to the other side. So: $\frac{dn}{n} = -\lambda\, dt$. Next we integrate both sides which gives: $\int \frac{dn}{n} = -\int \lambda\, dt$ and therefore: $\ln n = -\lambda t + c$ where c is the constant of integration. The exponential of both sides gives: $n = \exp(-\lambda t + c) = \exp(c)\exp(-\lambda t)$. Boundary conditions tells us at $t = 0$ $n = n_0$. So $n_0 = \exp(c)\exp(0)$ and therefore $\exp(c) = n_0$. Therefore $n = n_0 \exp(-\lambda t)$.

4.9 Radioactive dating

Suppose a particular isotope we'll call the parent decays into another, the daughter. At the start with only pure parent present there are P_0 nuclei, but after some time t a mixture of P parent and D daughter nuclei exists. From the previous section we can say that the number of parent nuclei P at time t is given by the expression $P(t) = P_0 \exp(-\lambda t)$.

There is of course no increase in the total number of nuclei (daughter + parent), so $P_0 = P + D$.

Substituting $(P + D)$ for P_0 we get: $P = (P + D)\exp(-\lambda t)$ which can be written $\frac{P}{(P+D)} = \exp(-\lambda t)$ or $\frac{(P+D)}{P} = \exp(\lambda t)$. This then becomes: $1 + \frac{D}{P} = \exp(\lambda t)$.

The significance of this is in its use in radioactive dating, telling us how long it has taken to reach the presently observed daughter to parent ratio so long as either λ or t_{half} are known.

4.9.1 Measuring the age of the oldest rocks

One way of aging igneous rocks, which contain uranium is to use the U-238 (Parent) to lead Pb-206 (Daughter) decay with a half-life of 4.5×10^9 years. The lead isotope is stable, so once formed it remains in the rock. The daughter to parent ratio is then found using chemical or mass spectrometric methods to determine the amount of the isotopes at the present time in the rock.

Example: Suppose we have found a lead to uranium (D/P) ratio of 5%. How old is the rock?

Firstly we find $\lambda = \frac{0.693}{t_{half}} = \frac{0.693}{4.5 \times 10^9 \times 365 \times 24 \times 3600} = 4.9 \times 10^{-18}\ \mathrm{s}^{-1}$.

Since $1 + \frac{D}{P} = \exp(\lambda t)$ we can say that $\ln\left(1 + \frac{D}{P}\right) = \lambda t$ and so $\ln(1.05) = 4.9 \times 10^{-18} t$. Therefore the age of the rock is $t = \frac{\ln(1.05)}{4.9 \times 10^{-18}} = \frac{0.0488}{4.9 \times 10^{-18}} = 9.96 \times 10^{15} s$ or 3.16×10^8 years.

Geologically, this rock cooled in the Carboniferous era, the vegetation dominated by giant ferns and the first land animals (amphibians) coming from the sea. This is quite young on the geological timescale: some of the oldest rocks in the USA are about 3.6×10^9 years old. Lunar rocks are older (there is little geological activity there to re-melt rocks, which resets the clock by remixing the isotopes).

4.9.2 Measuring the age (since death) of organic remains

Carbon has 14 isotopes ranging from C-8 to C-22. The most abundant and stable form is C-12. Only one of the unstable isotopes has a half-life longer than 25 mins. This is C-14, with a half-life of 5730 years. It turns out that C-14 is produced in the upper atmosphere, when neutrons interact with nitrogen:

$$n + {}^{14}_{7}N \rightarrow {}^{14}_{6}C + p.$$

Living things are mostly made of carbon, and get their carbon from carbon dioxide in the atmosphere. Therefore in a natural sample of carbon we would expect to find both stable C-12 and unstable C-14.

C-14 is unstable and undergoes β^- decay with a half-life of 5730 years:

$${}^{14}_{6}C \rightarrow {}^{14}_{7}N + e^-.$$

Unfortunately the daughter product nitrogen is a gas and will escape over time preventing a parent to daughter ratio from being established. So instead we measure the number of decays per second from a sample and compare the rate to a piece of modern carbon. Since the rate of decays is proportional to the number of parent atoms present, we can use the expression $n(t) = n_0 \exp(-\lambda t)$ where n_0 is the decay rate from modern carbon and $n(t)$ is the decay rate from the old sample. The age is then the time since the sample died and stopped refreshing its $^{14}C\ /\ ^{12}C$ ratio.

Since $\frac{n(t)}{n_0} = \exp(-\lambda t)$ then $-\lambda t = \ln\left(\frac{n(t)}{n_0}\right)$ and since $\lambda = \frac{0.693}{t_{half}}$ we can say that $t = -\frac{1}{\lambda}\ln\left(\frac{n(t)}{n_0}\right)$ and therefore that the age is $t = -\frac{t_{half}}{0.693}\ln\left(\frac{n(t)}{n_0}\right)$.

Example: Suppose a dried grain of wheat from a site of early human occupation gives 5 decays per second and a modern one of the same mass gives 30 decay counts per second. How old is the grain?

We know $t_{half} = 5730$ years and age $t = -\frac{t_{half}}{0.693}\ln\left(\frac{n(t)}{n_0}\right) = -\frac{5730 \times 365 \times 24 \times 3600}{0.693}\ln\left(\frac{5}{30}\right) = 4.67 \times 10^{11}\,\mathrm{s}$

The age is therefore 14,815 years.

C-14 dating can be used to date organic remains up to 50,000 years old.

Datasheet which will be attached to exam questions

MATHEMATICAL FORMULAE AND PHYSICAL CONSTANTS

Calculus

$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^x	e^x
$\log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\sinh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2+a^2}}$
$\cosh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2-a^2}}$
$\tanh^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2-x^2}$
uv	$u'v+uv'$
$\frac{u}{v}$	$\frac{u'v-uv'}{v^2}$

Definite Integrals

$$\int_0^\infty x^n e^{-ax}\mathrm{d}x = \frac{n!}{a^{n+1}}\left(n \geq 0 \text{ and } a>0\right)$$

$$\int_{-\infty}^{\infty} e^{-kx^2}\mathrm{d}x = \sqrt{\frac{\pi}{k}}$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

Integration by Parts

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int\frac{\mathrm{d}u}{\mathrm{d}x}v\mathrm{d}x$$

Spherical Polar Coordinates

$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta \quad \mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Physical Constants

charge on electron	$e = 1.60 \times 10^{-19}$ C
mass of electron	$m_e = 9.11 \times 10^{-31}$ kg = 0.511 MeV/c^2
mass of proton	$m_p = 1.673 \times 10^{-27}$ kg = 938.3 MeV/c^2
mass of neutron	$m_n = 1.675 \times 10^{-27}$ kg = 939.6 MeV/c^2
Planck constant	$h = 6.63 \times 10^{-34}$ J s
(Planck constant)/2π	$\hbar = 1.05 \times 10^{-34}$ J s
Boltzman constant	$k = 1.38 \times 10^{-23}$ J K^{-1}
	$= 8.62 \times 10^{-5}$ eV K^{-1}
speed of light in free space	$c = 3.00 \times 10^{8}$ m s^{-1}
permittivity of free space	$\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}
permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}
Avogadro constant	$N = 6.02 \times 10^{23}$ (g-mol)$^{-1}$
gas constant	$R = 8.32$ J (g-mol)$^{-1}$ K^{-1}
ideal gas volume (STP)	$V_0 = 22.4$ l (g-mol)$^{-1}$
gravitational constant	$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}
Rydberg constant	$R_\infty = 1.10 \times 10^{7}$ m^{-1}
Rydberg energy of hydrogen	$R_H = 13.6$ eV
Bohr radius	$a_0 = 0.529 \times 10^{-10}$ m
Bohr magneton	$\mu_B = 9.27 \times 10^{-24}$ J T^{-1}
fine structure constant	$\alpha \approx 1/137$
Wien displacement law constant	$b = 2.898 \times 10^{-3}$ m K
Stefan constant	$\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4}
radiation density constant	$a = 7.55 \times 10^{-16}$ J m^{-3} K^{-4}
mass of sun	$M_\odot = 1.99 \times 10^{30}$ kg
radius of sun	$R_\odot = 6.96 \times 10^{8}$ m
luminosity of sun	$L_\odot = 3.85 \times 10^{26}$ W
mass of earth	$M_\oplus = 6.0 \times 10^{24}$ kg
radius of earth	$R_\oplus = 6.4 \times 10^{6}$ m

Conversion Factors

1 u (atomic mass unit)	$= 1.66 \times 10^{-27}$ kg
	$= 931.5$ MeV/c^2
1 Å (angstrom) = 10^{-10} m	1 g (gravity) = 9.81 m s^{-2}
1 eV = 1.60×10^{-19} J	
1 atmosphere = 1.01×10^{5} Pa	1 year = 3.16×10^{7} s
1 parsec = 3.08×10^{16} m	1 astronomical unit = 1.50×10^{11}m

Binomial Expansion

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \ldots + \binom{n}{k}x^k y^{n-k} + \ldots + y^n$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots \left(|x| < 1\right)$$

Plane Polar Coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Trigonometry

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2\sin A \cos A$

$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

Series Expansions

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots$

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots$

$\log_e(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots (|x| < 1)$

Taylor's Series

$f(x+a) = f(a) + xf'(a) + \dfrac{x^2}{2!}f''(a) + \ldots$

Spherical Geometry

$\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$

$\cos a = \cos b \cos c + \sin b \sin c \cos A$

Vectors

$\underline{A}.\underline{B} = A_x B_x + A_y B_y + A_z B_z$

$\underline{A} \times \underline{B} = (A_y B_z - A_z B_y)\underline{i} + (A_z B_x - A_x B_z)\underline{j} + (A_x B_y - A_y B_x)\underline{k}$

$\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A}.\underline{C})\underline{B} - (\underline{A}.\underline{B})\underline{C}$

$\underline{A}.(\underline{B} \times \underline{C}) = \underline{B}.(\underline{C} \times \underline{A}) = \underline{C}.(\underline{A} \times \underline{B})$

Vector Calculus

$\nabla\phi = \dfrac{\partial\phi}{\partial x}\underline{i} + \dfrac{\partial\phi}{\partial y}\underline{j} + \dfrac{\partial\phi}{\partial z}\underline{k}$

$\nabla.\underline{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

$\nabla \times \underline{A} = \left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right)\underline{i} + \left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)\underline{j} + \left(\dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right)\underline{k}$

$\nabla.(\nabla\phi) = \nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

$\nabla \times (\nabla\phi) = 0$

$\nabla.(\nabla \times \underline{A}) = 0$

$\nabla \times (\nabla \times \underline{A}) = \nabla(\nabla.\underline{A}) - \nabla^2 \underline{A}$

Stirling's Formula

$\log_e N! \approx N\log_e N - N$

Series

$\sum_{k=1}^{n}\left[a + (k-1)d\right] = \dfrac{n}{2}\left[2a + (n-1)d\right]$

$\sum_{k=1}^{n} ar^{k-1} = a\dfrac{(1-r^n)}{1-r}$

Exponentials

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = \dfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$\sin\theta = \dfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

$\cosh\theta = \dfrac{1}{2}\left(e^{\theta} + e^{-\theta}\right)$

$\sinh\theta = \dfrac{1}{2}\left(e^{\theta} - e^{-\theta}\right)$